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Power-spectral Numbers

by

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Introduction, 1/3

Recall that modular arithmetic in $\mathbf{Z}_{12}$ is the set of equivalence classes of remainders modulo 12 endowed with operations of addition, subtraction, multiplication and, when possible, division. For example, it is easy to see that

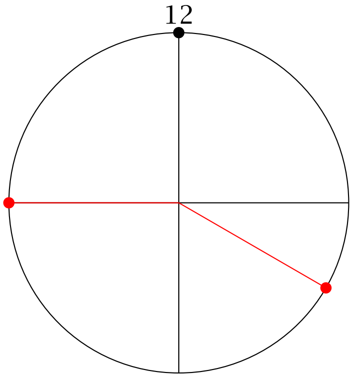
$$
\begin{aligned}
8 + 9 &= 5, \\
5 \cdot 7 &= 1, \\
2 \cdot 6 &= 0, \quad 3 \cdot 4 = 0.
\end{aligned}
$$

Consider

$$2^{57} = 8.$$

But what about 2^{57}? Since $2^2 = 4$, $2^3 = 8$, $2^4 = 4$, . . . , it is clear that $2^{\text{even}} = 4$ and $2^{\text{odd}} = 8$. Is there any way that operations in $\mathbf{Z}_{12}$ can be “simplified”?

Introduction, 2/3



$12 = (2)^2(3)$.

Spectral basis: $\{9, 4\}$.

Index=1.

Introduction, 3/3

Observe that, in $\mathbf{Z}_{12}$, we have

$$\begin{aligned} 9 + 4 &= 1, \\ 9 \cdot 4 &= 0, \\ 9^2 &= 9, \\ 4^2 &= 4. \end{aligned}$$

Furthermore, any $x \in \mathbf{Z}_{12}$ can be uniquely decomposed as

$$x = (x \bmod 4) \cdot 9 + (x \bmod 3) \cdot 4,$$

and

$$x^r = (x^r \mod 4) \cdot 9 + (x^r \mod 3) \cdot 4,$$

for all positive integers r. If x is invertible, then r can be negative as well.

The Spectral Basis Theorem

The elements 9 and 4 in $\mathbf{Z}_{12}$ comprise what is called the *spectral basis* for $\mathbf{Z}_{12}$, or for convenience, the spectral basis of 12. It is a fact that any integer n with at least two prime factors has a spectral basis.

Theorem 1

Let $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, $k > 1$, be a positive integer with at least two prime factors. Then there exist elements $s_1, s_2, \ldots, s_k$ of $\mathbf{Z}_n$ with the following properties:

$$s_1 + s_2 + \cdots + s_k = 1 \tag{1}$$

$$s_i^2 = s_i, 1 \le i \le k, \tag{2}$$

$$s_i s_j = 0, i \neq j, \tag{3}$$

$$x = (x^r \mod p_1^{e_1}) \cdot s_1 + \cdots + (x^r \mod p_k^{e_k}) \cdot s_k, (r \geq 0). \tag{4}$$

We call $\{s_1, s_2, \ldots, s_k\}$ the spectral basis of $\mathbf{Z}_n$, or, for convenience, the spectral basis of n.

Proof of the Spectral Basis Theorem, 1/2

- Define the map $\psi : \mathbf{Z} \to M$, $M := \mathbf{Z}_{p_1^{e_1}} \oplus \mathbf{Z}_{p_2^{e_2}} \oplus \cdots \oplus \mathbf{Z}_{p_k^{e_k}}$, by

$$\psi(x) = (\psi_1(x), \psi_2(x), \ldots, \psi_k(x)), \quad \psi_i(x) = x \mod p_i^{e_i}.$$

- Let us first find the image of ψ. Given $y = (\bar{y}_1, \ldots, \bar{y}_k)$, there exists $x \in \mathbf{Z}$ such that $\psi(x) = y$ if and only if $x \equiv \bar{y}_i \mod p_i^{e_i}$ for all $i = 1. \ldots, k$. Since the primary factors of n are pairwise relatively prime, by the Chinese Remainder Theorem the system of congruences has a solution, and so ψ is a ring epimorphism.
- Next, let us find the kernel of ψ. The kernel is all $x \in \mathbf{Z}$ such that $x \equiv 0 \mod p_i^{e_i}$ for all i, that is, if and only if x is divisible by $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. Consequently, the kernel of ψ is the ideal $n\mathbf{Z}$ and the induced map $\bar{\psi} : \mathbf{Z}/n\mathbf{Z} \to M$ is an isomorphism.

Proof of the Spectral Basis Theorem, 2/2

The direct sum $M := \mathbf{Z}_{p_1^{e_1}} \oplus \mathbf{Z}_{p_2^{e_2}} \oplus \cdots \oplus \mathbf{Z}_{p_k^{e_k}}$, has canonical projections $\pi_i : M \to \mathbf{Z}_{p_i^{e_i}}$ given by $\pi_i(n_1, \ldots, n_k) = n_i$ that satisfy

$$\begin{aligned} \pi_1 + \cdots + \pi_k &= \mathrm{Id}, \\ \pi_i^2 &= \pi_i, \\ \pi_i \pi_j &= 0, (i \neq j). \end{aligned}$$

What elements s_i of $\mathbf{Z}_n$ correspond to the projections π_i of M? Define $h_i := n/p_i^{e_i}$. Since $h_1, \ldots, h_k$ are pairwise relatively prime, there exists integers $a_1, \ldots, a_k$ in $\mathbf{Z}_n$ such that

$$a_1 h_1 + \cdots + a_k h_k = 1 \quad \text{in } \mathbf{Z}_n.$$

It can be shown that

$$s_i := a_i h_i = (h_i^{-1} \mod p_i^{e_i}) h_i$$

have the properties

$$\begin{aligned} s_1 + \cdots + s_k &= 1, \\ s_i^2 &= s_i, \\ s_i s_j &= 0, (i \neq j). \end{aligned}$$

Power-spectral numbers

Definition 2

A positive integer is *power-spectral* if its spectral basis consists of primes or powers.

Examples 3

1. $\{3, 4\}$ is the spectral basis for 6.
2. $\{9, 4\}$ is the spectral basis for 12.
3. $\{7, 8\}$ is the spectral basis for 14.
4. $\{9, 16\}$ is the spectral basis for 24.
5. $\{15^2, 2^6\}$ is the spectral basis for $288 = (2)^5(3)^2$.
6. $\{15^2, 20^2, 24^2\}$ is the spectral basis for $600 = (2)^3(3)(5)^2$.

Mersenne I, 1/2

Theorem 4

The number $2p^k$ *has spectral basis* $\{p^k, p^k + 1\}$.

Corollary 5

The number $2M_p$ *has spectral basis* $\{M_p, 2^p\}$.

Examples 6

1. $\{3, 2^2\}$ is the spectral basis for $2 \cdot 3$.
2. $\{7, 2^3\}$ is the spectral basis for $2 \cdot 7$.
3. $\{31, 2^5\}$ is the spectral basis for $2 \cdot 31$.
4. $\{127, 2^7\}$ is the spectral basis for $2 \cdot 127$.

Mersenne I, 2/2

Theorem 7

Let M_p be a Mersenne prime with Mersenne exponent p. Then the following numbers are power-spectral.

1. $2M_p$ *has spectral basis* $\{M_p, 2^p\}$ *or, equivalently,* $\{M_p, M_p + 1\}$.
2. $2^p M_p$ *has spectral basis* $\{M_p^2, 2^p\}$ *or, equivalently,* $\{M_p^2, M_p + 1\}$.
3. $2^{p+1} M_p$ *has spectral basis* $\{M_p^2, 2^{2p}\}$ *or, equivalently,* $\{M_p^2, (M_p + 1)^2\}$
4. $2^{2p+1} M_p^2$ *has spectral basis* $\{M_p^2 (M_p + 2)^2, (M_p^2 - 1)^2\}$.

Fermat I, 1/1

It is easily shown that $2^a + 1$ can be prime if and only if a is a power of 2. The number $F_i = 2^{2^i} + 1$, $i \geq 0$, is called a *Fermat number* and a *Fermat prime* when it is prime. The only known Fermat primes are $F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$.

Theorem 8

If $F_i = 2^{f_i} + 1$ is a Fermat prime with exponent $f_i = 2^i$, $i \geq 0$, then

1. $2^{f_i} F_i$ *has spectral basis* $\{F_i, 2^{2f_i}\}$.
2. $2^{f_i+1} F_i$ *has spectral basis* $\{F_i^2, 2^{2f_i}\}$.
3. $2^{2f_i+1} F_i^2$ *has spectral basis* $\{(F_i - 2)^2 F_i^2, (F_i^2 - 1)^2\}$.

Cyclotomic primes, 1/3

Consider the number $20439 = 3^3 \cdot 757$. Let us verify that $\{757, 3^9\}$ is the spectral basis for 20439. Clearly,
$757 + 3^9 = 20440 \equiv 1 \mod 20439$ and $757 \cdot 3^9 \equiv 0 \mod 20439$.
Further,

$$\begin{aligned} 757^2 - 757 &= 757 \cdot 756 = 757 \cdot 2^2 \cdot 3^3 \cdot 7 \\ &= 2^2 \cdot 7 \cdot (3^3 \cdot 757) \equiv 0 \mod 20439. \\ (3^9)^2 - 3^9 &= 3^9(3^9 - 1) = 3^9 \cdot 2 \cdot 13 \cdot 757 \\ &= 2 \cdot 3^6 \cdot 13 \cdot (3^3 \cdot 757) \equiv 0 \mod 20439. \end{aligned}$$

Are 757 and 3^9 related? The key is the decomposition of the identity.

Cyclotomic primes, 2/3

$$
\begin{aligned}
757 + 3^9 &= 3^3 \cdot 757 + 1 \\
3^9 - 1 &= 3^3 \cdot 757 - 757 \\
3^9 - 1 &= (3^3 - 1)(757) \\
757 &= \frac{3^9 - 1}{3^3 - 1}
\end{aligned}
$$

Definition 9

The number $\Phi_{r^e}(p) = \dfrac{p^{r^e} - 1}{p^{r^{e-1}} - 1}$, where p and r are primes and $e \geq 1$, when prime, is called a *cyclotomic prime*.

NOTE: $\Phi_{r^e}(x) = \dfrac{x^{r^e} - 1}{x^{r^{e-1}} - 1}$ can be prime when x is composite but we are only interested in the case when x is prime.

Cyclotomic primes, 3/3

Theorem 10

The number $p^{r^{e-1}}\Phi_{r^e}(p)$ *has spectral basis* $\{\Phi_{r^e}(p), p^{r^e}\}$, *where* $\Phi_{r^e}(p)$ *is a cyclotomic prime.*

Proof.

The decomposition of the identity follows from the requirement that $\Phi_{r^e}(p)$ is prime. Let's verify the projection property for $q = \Phi_{r^e}(p)$. Observe that

$$\begin{aligned} q^2 - q = q(q-1) &= q\left(\frac{p^{r^e}-1}{p^{r^{e-1}}-1} - 1\right) \\ &= q\left(\frac{p^{r^e}-p^{r^{e-1}}}{p^{r^{e-1}}-1}\right) \\ &= p^{r^{e-1}}q\left(\frac{p^{r^e-r^{e-1}}-1}{p^{r^{e-1}}-1}\right) \\ &\equiv 0 \mod p^{r^{e-1}}q. \end{aligned}$$

□

Exercise: $(p^{r^e})^2 \equiv p^{r^e} \mod p^{r^{e-1}}q.$

Power-spectral numbers $9p^{2s}q^{2t}$, 1/3

Of natural interest are primes solutions to $q^t = 2p^s \pm 1$ with $s, t \geq 1$. For example, **Sophie-Germain primes** are primes of the form $q = 2p + 1$ and **Cunningham primes** are of the form $q = 2p - 1$. It is open question whether or not there are infinitely many primes of the form $q = 2p \pm 1$.

Theorem 11 (Pell equation)

The equations $x^2 - 2y^2 = \pm 1$ have infinitely many integer solutions. The only prime solution to $x^2 - 2y^2 = 1$ is $(x, y) = (3, 2)$. The only prime solutions to $x^2 - 2y^2 = -1$ known so far are

$$
\begin{aligned}
(7)^2 &= 2(5)^2 - 1 \\
(41)^2 &= 2(29)^2 - 1 \\
(63018038201)^2 &= 2(44560482149)^2 - 1 \\
(19175002942688032928599)^2 &= 2(13558774610046711780701)^2 - 1
\end{aligned}
$$

Power-spectral numbers $9p^{2s}q^{2t}$, 2/3

Theorem 12 (Ljjungren, 1942)

The only positive integer solutions to $y^2 = 2x^4 - 1$ *are* $(x, y) = (1, 1)$ *and* $(13, 239)$*, and the only prime solution is* $(13, 239)$*.*

Theorem 13 (Crescenzo, 1975)

The only solutions to $q^t = 2p^s \pm 1$, $s, t > 1$*, for primes* p *and* q *occur only for* $(s, t) = (2, 2)$ *and* $(4, 2)$*.*

Theorem 14 (Solutions to $q^t = 2p^s \pm 1$)

The only prime solutions to $q^t = 2p^s \pm 1$, $s, t \geq 1$*, occur for* $(s, 1)$, $(1, t)$, $(2, 2)$*, and* $(4, 2)$*.*

Power-spectral numbers $9p^{2s}q^{2t}$, 3/3

Theorem 15

Suppose $q^t = 2p^s \pm 1$ has prime solutions, $p, q \neq 3$, for some positive integers s and t. Then $9p^{2s}q^{2t}$ has spectral basis

$$\{p^{2s}q^{2t}, 4(p^{2s} - 1)^2, 16(p^2 \pm 1)p^{2s}\}.$$

Definition 16 (Ljjungren's number)

Ljjungren's number is defined to be the power-spectral number

$$3^2(13)^8(239)^4 = 2395415920687164 1449.$$

It is the unique power-spectral number of the form $9p^8q^4$ where p and q are prime.

Mersenne II, 1/2

Theorem 17

Let M_p is a Mersenne prime with Mersenne exponent $p > 2$. Then

1. $2^{2p-1} \cdot 3 \cdot M_p^2$ *has power-spectral basis*

$$\left\{M_p^2(M_p+2)^2, M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}$$

of index 2.

2. $2^{2p} \cdot 3 \cdot M_p^2$ *has power-spectral basis*

$$\left\{M_p^2(M_p+2)^2, M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}.$$

3. $2^{2p+1} \cdot 3 \cdot M_p^2$ *has power-spectral basis*

$$\left\{M_p^2\left(M_p+2\right)^2, 4M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}.$$

The numbers 1 and 2 comprise an isospectral pair. See 22.

Mersenne II, 1/2

Theorem 18

Let M_p be a Mersenne prime with Mersenne exponent $p > 2$. Then

1. $2^{2p-3} \cdot 3^2 \cdot M_p^2$ *has power-spectral basis*

$$\left\{M_p^2(M_p+2)^2, \frac{1}{4}M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}$$

   of index 2.

2. $2^{2p-2} \cdot 3^2 \cdot M_p^2$ *has power-spectral basis*

$$\left\{M_p^2(M_p+2)^2, \frac{1}{4}M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}.$$

3. $2^{2p+1} \cdot 3^2 \cdot M_p^2$ *has power-spectral basis*

$$\left\{M_p^2(M_p+2)^2, 16M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}.$$

Furthermore, the numbers 1 and 2 comprise an isospectral pair.
See 22.

Fermat II, 1/2

Theorem 19

Let F_i be a Fermat prime with exponent $f_i = 2^i$. Then the following numbers are power-spectral.

1. $2^{2f_i-1} \cdot 3 \cdot F_i^2$ *has power-spectral basis*

$$\{(F_i-2)^2F_i^2, (F_i-1)^2 \cdot F_i^2, (F_i^2-1)^2\}.$$

   with index 2.

2. $2^{2f_i} \cdot 3 \cdot F_i^2$ *has power-spectral basis*

$$\{(F_i-2)^2F_i^2, (F_i-1)^2F_i^2, (F_i^2-1)^2\}.$$

3. $2^{2f_i+1} \cdot 3 \cdot F_i^2$ *has power-spectral basis*

$$\{(F_i-2)^2F_i^2, 4(F_i-1)^2 \cdot F_i^2, (F_i^2-1)^2\}.$$

Furthermore, 1 and 2 form an isospectral pair. See 22.

Fermat II, 2/2

Theorem 20

Let F_i be a Fermat prime with Fermat exponent $f_i = 2^i$. Then

1. $2^3 \cdot 9 \cdot 5^2$ *has power-spectral basis*

$$\left\{3^2 5^2, 2^3 5^3, 2^6 3^2\right\}.$$

2. $2^{2f_i - 3} \cdot 9 \cdot F_i^2$ *has power-spectral basis*

$$\left\{(F_i - 2)^2 F_i^2, \frac{1}{4}(F_i - 1)^2 F_i^2, (F_i^2 - 1)^2\right\}.$$

   with index 2.

3. $2^{2f_i - 2} \cdot 9 \cdot F_i^2$ *has power-spectral basis*

$$\left\{(F_i - 2)^2 F_i^2, \frac{1}{4}(F_i - 1)^2 F_i^2, (F_i^2 - 1)^2\right\}.$$

4. $2^{2f_i + 1} \cdot 9 \cdot F_i^2$, *has power-spectral basis*

$$\left\{(F_i - 2)^2 F_i^2, 16(F_i - 1)^2 F_i^2, (F_i^2 - 1)^2\right\}.$$

Furthermore, the numbers of Theorem 2 and Theorem 3 form an isospectral pair for $i = 2, 3, 4$. See 22.

Isospectral chains, 1/3

The pair $\{84, 42\}$ both have the same spectral basis, namely, $\{21, 28, 36\}$. Two numbers will be called *isospectral* if they have the same spectral basis. Let's look at the decomposition of the identity.

$$21 + 28 + 36 = 2 \cdot 42 + 1 \equiv 1 \mod 42,$$
$$21 + 28 + 36 = 1 \cdot 84 + 1 \equiv 1 \mod 84.$$

We say that 42 has index 2 and that 84 has index 1 and that $\{84, 42\}$ comprise an *isospectral pair*.

Definition 21 (Isospectral pair)

An *isospectral pair* is a pair of integers $\{n_1, n_2\}$ such that $n_1 = 2n_2$, both have the same spectral basis, and of index 1 and 2, respectively.

Maximal isopectral chains of length 2.

n_1	n_1 factored	
84	$(2)^2(3)(7)$	$\{21, 28, 36\}$
228	$(2)^2(3)(19)$	$\{57, 76, 96\}$
280	$(2)^3(5)(7)$	$\{105, 56, 120\}$
340	$(2)^2(5)(17)$	$\{85, 136, 120\}$

Isospectral chains, 2/3

Definition 22

An *isospectral chain* of length k is defined to be a finite sequence of pairwise isospectral numbers $n_1, \dots, n_k$, such that n_i has index i and

$$n_1 + 1 = 2n_2 + 1 = \cdots = kn_k + 1,$$

or, equivalently,

$$n_1 = 2n_2 = \cdots = kn_k.$$

It will be assumed that the chain length k is maximal, that is, $n_1/(k+1)$ is not isospectral with n_1.

Isospectral chains, 3/3

Maximal isopectral chains of length 3.

n_1	n_1 factored	
10980	$(2)^2(3)^2(5)(61)$	$\{2745, 2440, 2196, 3600\}$
35280	$(2)^4(3)^2(5)(7)^2$	$\{11025, 7840, 7056, 9360\}$
36180	$(2)^2(3)^3(5)(67)$	$\{9045, 10720, 7236, 9180\}$
43380	$(2)^2(3)^2(5)(241)$	$\{10845, 9640, 8676, 14220\}$

Maximal isopectral chains of length 4.

n_1	n_1 factored	
488880	$(2)^4(3)^2(5)(7)(97)$	$\{91665, 108640, 97776, 69840, 120960\}$
1525680	$(2)^4(3)^2(5)(13)(163)$	$\{286065, 339040, 305136, 352080, 243360\}$
2870280	$(2)^3(3)^2(5)(7)(17)(67)$	$\{358785, 637840, 574056, 410040, 675360, 214200\}$
4930272	$(2)^5(3)^2(17)(19)(53)$	$\{1078497, 1095616, 1160064, 1037952, 558144\}$

Isotropic numbers, 1/4

- Recall that $42 = 2 \cdot 3 \cdot 7$ is the first number of index 2 with spectral basis $\{21, 28, 36\}$. Since $\{1 \cdot 21, 2 \cdot 14, 6 \cdot 6\}$, we call $\{1, 2, 6\}$ the *spectral coefficients* of 42.
- Consider the product of twin primes $3 \cdot 5 = 15$, with spectral basis $\{10, 6\}$. Observe that $10 = 2 \cdot 5$ and $6 = 3 \cdot 2$ so that the spectral coefficients of 15 are $\{2, 2\}$.

Definition 23 (Isotropic number)

A number is *isotropic* if all its spectral coefficients are equal.

Theorem 24

The product of twin primes is isotropic.

Proof.

Let p and $q = p + 2$ be prime. Then $aq + ap = pq + 1$ so that $a = (pq + 1)/(p + q) = (p^2 + 2p + 1)/(2p + 2) = (p + 1)^2/(2(p + 1)) = (p + 1)/2$. It can shown that $\{aq, ap\}$ is in fact the spectral basis for pq. □

Isotropic numbers, 2/4

Theorem 25

If p and q are primes or prime powers, and if

$$a = (pq + 1)/(p + q)$$

is an integer, then pq is isotropic with spectral coefficient a.

Powerful isotropic numbers with two factors

1728	$(2)^6(3)^3$	$\{513, 1216\}$
675	$(3)^3(5)^2$	$\{325, 351\}$
7092899	$(11)^3(73)^2$	$\{5675385, 1417515\}$
7138196909	$(29)^3(541)^2$	$\{6589127353, 549069557\}$

Isotropic numbers, 3/4

Theorem 26 (Isotropic number theorem)

Let $n = P_1 \cdots P_k$ be a product of distinct primes or prime powers. Let $\bar{P}_i = n/P_i$ and suppose that

$$a = (n+1)/(\bar{P}_1 + \cdots \bar{P}_k)$$

is an integer. Then n is isotropic with spectral coefficient a and spectral basis $\{a\bar{P}_1, \ldots, a\bar{P}_k\}$.

Isotropic numbers with more than two factors

n		a
30	$(2)(3)(5)$	1
429	$(3)(11)(13)$	2
858	$(2)(3)(11)(13)$	1
861	$(3)(7)(41)$	2
1722	$(2)(3)(7)(41)$	1
2300	$(2)^2(5)^2(23)$	3

Isotropic numbers, 4/4

Isotropic numbers of immediate interest are those with $a = 1$, called *cancelable*, since the spectral basis is found by deletion of prime factors.

Isotropic numbers $a = 1$

30	$(2)(3)(5)$	1
858	$(2)(3)(11)(13)$	1
1722	$(2)(3)(7)(41)$	1
66198	$(2)(3)(11)(17)(59)$	1

A search on the Online Encyclopedia of Integer Sequences, `https://oeis.org/`, reveals the following:

A007850 **Giuga numbers:** composite numbers n such that p divides $n/p - 1$ for every prime divisor p of n.

$$30, 858, 1722, 66198, 2214408306, 24423128562, \ldots$$

It is easy to show that ever Giuga number is cancelative.

Conjecture 1

A number is cancelative if and only if it is Giuga.

Fibonacci 1/2

Recall that the Fibonacci sequence is defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$. Since $F_m | F_n$ whenever $m | n$, F_n can be prime only when n is prime.

Lemma 27

Let p be a prime such that F_p is prime. Then

$$F_p \equiv \left(\frac{5}{p}\right) \pmod{p},$$

where $(5|p)$ is the Legendre symbol defined by

$$\left(\frac{5}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1, 4 \pmod{5}; \\ -1 & \text{if } p \equiv 2, 3 \pmod{5}. \end{cases}$$

Fibonacci 1/2

Theorem 28

Let $p \neq 5$ be a prime such that F_p is prime. Then pF_p has spectral basis

$$\begin{aligned} \{F_p, pF_p - F_p + 1\} &\quad \textit{whenever } p \equiv 1,4 \pmod 5, \\ \{(p-1)F_p, F_p + 1\} &\quad \textit{whenever } p \equiv 2,3 \pmod 5. \end{aligned}$$

Lucas 1/1

Recall that the Lucas sequence is defined recursively by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$, $n \geq 2$. Since $L_m | L_n$ whenever $m|n$ and n/m is odd, L_n can be prime only when n is prime or a power of 2.

Lemma 29

1. *Let p be a prime such that L_p is prime. Then $L_p \equiv 1 \bmod p$.*
2. *If L_{2^m} is prime, then $L_{2^m} \equiv -1 \bmod p$.*

Theorem 30

1. *If p is a prime such that L_p is prime, then pL_p has spectral basis $\{L_p, pL_p - L_p + 1\}$.*
2. *If L_{2^m} is prime, then $2^m L_{2^m}$ has spectral basis $\{(2^m - 1)L_{2^m}, L_{2^m} + 1\}$.*

NOTE: L_{2^m} is known to be prime only for $m = 1, 2, 3, 4$, just like the Fermat primes. ■