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Phoenix and Vampire Numbers

by

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Phoenix Numbers

Definition 1

A positive integer n is *k-Phoenix in base b* if the digits of n are split at the kth digit, $n = a|b$, then $n = a_r \cdot b_r$, where a_r, b_r are the numbers in base b with the digits of a and b are reversed.

Example 2

688 is 1-Phoenix in base 10 since $8 \cdot 86 = 688$.

I first searched for 1-Phoenix numbers in base 12. Here are some examples. Note that in base 12 the digits are $0, 1, \ldots, 9$, X, E.

3.7.6	3.7.8.0.4.3.7.6
3.E.7.6	3.7.8.0.0.4.3.7.6
3.E.E.7.6	3.7.8.0.0.0.4.3.7.6
3.E.E.E.7.6	3.7.8.0.4.3.7.8.0.4.3.7.6
3.7.8.4.3.7.6	3.7.8.0.4.3.7.8.0.4.3.7.8.0.4.3.7.6
3.7.8.4.3.7.8.4.3.7.6	3.7.8.0.0.4.3.7.8.0.0.4.3.7.8.0.0.4.3.7.6
3.7.8.4.3.7.8.4.3.7.8.4.3.7.6	3.E.7.8.0.4.3.E.7.6
3.E.7.8.4.3.E.7.6	3.E.E.7.8.0.4.3.7.8.0.4.3.E.E.7.6
3.E.E.7.8.4.3.E.E.7.6	3.E.E.7.8.0.4.3.7.8.0.4.3.E.E.7.6
3.E.E.E.7.8.4.3.7.8.4.3.E.E.E.7.6	3.E.E.7.8.0.0.4.3.7.8.0.0.0.4.3.E.E.7.6

Theorem 3

The following numbers are 1-Phoenix in base 12.

1. $3.[\mathsf{E}].7.6$
2. $3.[7.8.4.3].7.6$
3. $3.[\![\mathsf{E}]\!].[7.8.4.3].[\![\mathsf{E}]\!].7.6$
4. $3.[7.8.[0].4.3].7.6$
5. $3.[\![\mathsf{E}]\!].[7.8.[0].4.3].[\![\mathsf{E}]\!].7.6$

Notation

- $[D]$ means zero or more occurrences of the digits D.
- $[D_1[D_3]D_2]$ is specified from the inside out, where $D = D_1D_2$ is itself a repeating pattern.
- All patterns within $[\![\,]\!]$ represent the same number of instances, even if the double brackets contain different patterns.

Multiplication Table in Base 12 for Theorem 3.5

(3)	(5)	⟦(5)⟧	(1)	(2)	(0)	[(0)]	(4)	(3)	(5)	⟦(5)⟧	(1)	
	7.	⟦E⟧.	E.	[3.	4.	[0].	0.	8.	7].	⟦E⟧.	E.	3
												6
3.	E.	⟦E⟧.	[7.	8.	0.	[0].	4.	3].	E.	⟦E⟧.	7.	6

$$6 \cdot 3 = 1.6$$
$$6 \cdot \mathsf{E} + 1 = 5.7$$
$$6 \cdot \mathsf{E} + 5 = 5.\mathsf{E}$$
$$6 \cdot 7 + 5 = 3.\mathsf{E}$$
$$6 \cdot 4 = 2.0$$
$$6 \cdot 3 + 2 = 1.8$$

1-Phoenix Numbers With Three Digits

A 1-Phoenix number must have at least three digits. Here is a table of 3-digit 1-Phoenix numbers in bases up to 12.

4	1.2.3
6	1.2.4, 1.3.3
8	1.2.5, 1.4.3
9	1.3.4
10	1.2.6, 1.5.3
12	1.2.7, 1.3.5, 1.4.4, 1.6.3

Do you see the pattern?

Base $b = de$ Theorem

Theorem 4

If $d, e > 1$, then

$$1.e.(d+1)$$

is 1-Phoenix in base $b = de$.

Proof.

(1)		
	$e.$	1
		$d+1$
$1.$	$e.$	$d+1$

Base $b = de$.

$(d+1)e = de + e = 1.e$ in base $b = de$. □

1-Phoenix Powers

A *1-Phoenix power* in base b a 1-Phoenix number in base b that is also a power. Here is a table of 1-Phoenix powers of the form $n = 1.e.(d+1)$ in base $b = de$.

b	n	n_b	$(m_b)^p$	
4	27	1.2.3	$(3)^3$	(C)
12	196	1.4.4	$(1.2)^2$	$(S1)$
21	512	1.3.8	$(8)^3$	(C)
22	729	1.11.3	$(1.5)^2$	$(S2)$
45	2704	1.15.4	$(1.7)^2$	$(S2)$
48	2601	1.6.9	$(1.3)^2$	$(S1)$
56	3375	1.4.15	$(15)^3$	(C)
76	7225	1.19.5	$(1.9)^2$	$(S2)$
115	13824	1.5.24	$(24)^3$	(C)
115	15876	1.23.6	$(1.11)^2$	$(S2)$
120	15376	1.8.16	$(1.4)^2$	$(S1)$
120	16807	1.20.7	$(7)^5$	

1-Phoenix Squares

b	$1.e.d+1$	$(1.x)^2$	$(x^2)_b$
12	1.4.4	1.2^2	4
22	1.11.3	3^6	1.3
45	1.15.4	1.7^2	1.4
48	1.6.9	1.3^2	9
76	1.19.5	1.9^2	1.5
115	1.23.6	1.11^2	1.6
120	1.8.16	1.4^2	16
162	1.27.7	1.13^2	1.7
168	1.56.4	1.26^2	4.4
217	1.31.8	1.15^2	1.8
*218	1.109.3	1.49^2	11.3
240	1.10.25	1.5^2	25
*627	1.209.4	1.97^2	15.4
*1364	1.341.5	1.161^2	19.5

1-Phoenix Squares Theorem

Not all squares have a formula. However, we have the following.

Theorem 5

1. *If* $x \geq 2$, *then* $1.2x.x^2$ *is 1-Phoenix in base* $b = 2x(x^2 - 1)$ *and*

$$1.2x.x^2 = (1.x)^2 = x^2(2x^2 - 1)^2.$$

2. *Given odd* $x \geq 5$, *let* $d = (x - 1)/2$ *and* $e = 2x + 1$. *Then* $1.e.(d + 1)$ *is 1-Phoenix in base* $b = de$ *and*

$$1.e.(d + 1) = (1.x)^2 = \left(\frac{x + 1}{2}\right)^2 (2x - 1)^2.$$

1-Phoenix Squares

d	e	b	$1.e.(d+1)$	$(1.x)^2$	x_b^2
2	109	218	1.109.3	1.49^2	11.3
3	209	627	1.209.4	1.97^2	15.4
4	341	1364	1.341.5	1.161^2	19.5

Theorem 6

The number $1.16n^2 - 12n + 1.n$, $n \geq 3$, *is 1-Phoenix in base* $b = (n-1)(16n^2 - 12n + 1)$, *and*

$$1.16n^2 - 12n + 1.n = (1.(8n^2 - 8n + 1))^2.$$

Proof.

Observe from the table that x_b^2 is of the form $4n - 1.n$ so $d = n - 1$. Furthermore, the sequence $49, 97, 161, \ldots$ is A069129 (**Centered 16-gonal numbers**) : $a(n) = 8n^2 - 8n + 1$. Therefore, $e = 2a(n) + (4n-1) = 16n^2 - 12n + 1$ and $b = (n-1)(16n^2 - 12n + 1)$. It can be shown that $1.16n^2 - 12n + 1.n = (1.(8n^2 - 8n + 1))^2$. □

1-Phoenix Cubes

b	n	n_b	cube	
4	27	1.2.3	3^3	
21	512	1.3.8	8^3	
22	729	1.11.3	9^3	$\star$
56	3375	1.4.15	15^3	
115	13824	1.5.24	24^3	
204	42875	1.6.35	35^3	
273	85184	1.39.8	44^3	$\star$
329	110592	1.7.48	48^3	
496	250047	1.8.63	63^3	
711	512000	1.9.80	80^3	
819	681472	1.13.64	88^3	$\star$
980	970299	1.10.99	99^3	
1309	1728000	1.11.120	120^3	
1704	2924207	1.12.143	143^3	

The e^2 Theorem for Phoenix Powers I

It is easy to observe that many 1-Phoenix numbers of the form $1.e.(d+1)$ are cubes of the form $(d+1)^3$. Let's consider powers in general.

$$\begin{aligned}(ed)^2 + e(ed) + (d+1) &= (d+1)^n \\ e^2 d(d+1) &= (d+1)^n - (d+1) \\ e^2 &= \frac{(d+1)^{n-1} - 1}{d}\end{aligned}$$

Theorem 7

If $e^2 = ((d+1)^{n-1} - 1)/d$, $n \geq 3$, *then* $1.e.(d+1)$ *is 1-Phoenix in base* $b = de$, *and* $1.e.(d+1) = (d+1)^n$. *If* $n = 3$, *then* $e^2 = d + 2$ *so* $d = e^2 - 2$, $b = e(e^2 - 2)$, *and*

$$1.e.(e^2 - 1) = (e^2 - 1)^3.$$

The only exceptions to $n = 3$ out to base $b = 10^6$ are

22	729	1.11.3	9^3
273	85184	1.39.8	44^3
819	681472	1.13.64	88^3
28236	827936019	1.1086.27	939^3

The e^2-Theorem for Phoenix Powers II

Observe that $e^2 = ((d+1)^{n-1} - 1)/d$ can be written as

$$\frac{x^m - 1}{x - 1} = e^2, \quad x = d + 1, \quad m = n - 1. \tag{1}$$

In 1943 Ljunggren showed that the only solutions to (1) with $m > 2$ are

$\frac{7^4 - 1}{7 - 1} = 20^2$	$\frac{3^5 - 1}{3 - 1} = 11^2$
$1.20.7 = 7^5$	$1.11.3 = 3^6$
$b = 120$	$b = 22$

FYI: $\frac{18^3 - 1}{18 - 1} = 7^3$ is the only known solution to

$$\frac{x^m - 1}{x - 1} = y^p, \quad m, p > 2.$$

It is conjectured that there are no other solutions.

1-Phoenix Numbers with Prime Base I

Clearly, the number $1.e.(d+1)$ is 1-Phoenix in the composite base $b = de$. What about prime number bases?

3	1.1.2.2	
3	1.1.0.2.2	
3	1.1.0.0.2.2	$\star$
5	1.3.2.0.2.4	$\star$
7	1.5.4.0.2.6	$\star$
11	1.9.8.0.2.10	$\star$
13	1.11.10.0.2.12	$\star$
17	1.7.2.10	
17	1.11.9.3.8	
17	1.15.3.6.5.6	
17	1.15.14.0.2.16	$\star$
19	1.7.4.6	
19	1.17.16.0.2.18	$\star$
23	1.9.7.18.1.18	
23	1.21.20.0.2.22	$\star$

1-Phoenix Numbers with Prime Base II

The $\star$ numbers appear to be following a pattern.

3	1.1.0.0.2.2
5	1.3.2.0.2.4
7	1.5.4.0.2.6
11	1.9.8.0.2.10
13	1.11.10.0.2.12
17	1.15.14.0.2.16
19	1.17.16.0.2.18
23	1.21.20.0.2.22

Universal 1-Phoenix Numbers I.a

Theorem 8

If b *is any base greater than 2, let* $\mathsf{B} = b - 1$, $\mathsf{C} = b - 2$ *and* $\mathsf{D} = b - 3$. *Then*

$$1.\mathsf{C}.\mathsf{D}.0.2.\mathsf{B}$$

is 1-Phoenix in base b.

		(D)	(D)		
	2.	0.	D.	C.	1
					B
1.	C.	D	0.	2.	B

Base b.

$$\mathsf{B} \cdot \mathsf{C} = (b-1)(b-2) = b^2 - 3b + 2 = (b-3)b + 2 = \mathsf{D}.2$$

$$\mathsf{B} \cdot \mathsf{D} + \mathsf{D} = \mathsf{D} \cdot b = \mathsf{D}.0$$

$$\mathsf{B} \cdot 2 = 2b - 2 = b + (b-2) = 1.\mathsf{C}$$

Universal 1-Phoenix Numbers I.b

Theorem 9

If $b \geq 3$ is any base, let B $= b - 1$, C $= b - 2$ *and* D $= b - 3$*. Then*

$$1.\mathsf{C}.\mathsf{D}.0.2.\mathsf{B}$$

is 1-Phoenix in base b. Furthermore,

1. 1.C.D.[0.2.1.0.C.D].0.2.B *is 1-Phoenix in base b.*
2. 1.C.D.[B.1.2.B.D.C].0.2.B *is 1-Phoenix in base b.*

Therefore, there are infinitely many 1-Phoenix numbers in any base b greater than two.

		(D)	(D)		(1)	(1)		(D)	(D)		
	2.	0.	[D.	C.	0.	1.	2.	0]	D.	C	1
											B
1.	C.	D.	[0.	2.	1.	0.	C.	D].	0.	2.	B

		(D)	(D)	(C)	(1)	(1)	(C)	(D)	(D)		
	2.	0.	[C.	D.	B.	2.	1.	B].	D.	C	1
											B
1.	C.	D.	[B.	1.	2.	B.	D.	C].	0.	2.	B

Universal 1-Phoenix Numbers I.c

Theorem 10

If b is any base greater than 2, let $\mathsf{B} = b-1$, $\mathsf{C} = b-2$ and $\mathsf{D} = b-3$. Then

$$1.\mathsf{C}.\mathsf{D}.0.2.\mathsf{B}$$

is 1-Phoenix in base b. Furthermore,

1. $1.\mathsf{C}.\mathsf{D}.[0.2.1.0.\mathsf{C}.\mathsf{D}].0.2.\mathsf{B}$ *is 1-Phoenix in base b.*
2. $1.\mathsf{C}.\mathsf{D}.[\mathsf{B}.1.2.\mathsf{B}.\mathsf{D}.\mathsf{C}].0.2.\mathsf{B}$ *is 1-Phoenix in base b.*
3. *Let $\mathsf{M}_0, \mathsf{M}_1, \ldots, \mathsf{M}_k$ be instances of $[0.2.1.0.\mathsf{C}.\mathsf{D}]$ and let $\mathsf{N}_0, \ldots, \mathsf{N}_k$ be instances of $[\mathsf{B}.1.2.\mathsf{B}.\mathsf{D}.\mathsf{C}]$. Then*

$$1.\mathsf{C}.\mathsf{D}.\left[\mathsf{M}_k.\mathsf{N}_k \ldots \mathsf{M}_1.\mathsf{N}_1.\mathsf{M}_0.\mathsf{N}_1.\mathsf{M}_1 \ldots \mathsf{N}_k.\mathsf{M}_k\right].0.2.\mathsf{B}$$

and

$$1.\mathsf{C}.\mathsf{D}.\left[\mathsf{N}_k.\mathsf{M}_k \ldots \mathsf{N}_1.\mathsf{M}_1.\mathsf{N}_0.\mathsf{M}_1.\mathsf{N}_1 \ldots \mathsf{M}_k.\mathsf{N}_k\right].0.2.\mathsf{B}$$

are 1-Phoenix in base b.

Universal k-Phoenix Numbers, $k \geq 2$, I.a

Consider the following table in base 12.

k	
2	1.0.E.0.1.E.E.<u>1.1</u>
3	1.0.0.E.E.0.0.0.1.0.E.E.E.E.<u>1.0.1</u>
4	1.0.0.0.E.E.E.0.0.0.0.0.1.0.0.E.E.E.E.E.E.<u>1.0.0.1</u>
5	1.0.0.0.0.E.E.E.E.0.0.0.0.0.0.0.1.0.0.0.E.E.E.E.E.E.E.E.<u>1.0.0.0.1</u>

Universal k-Phoenix Numbers, $k \geq 2$, I.b

Let $\mathsf{B} = b - 1$ be the largest digit in base b. Here is the pattern:

k	
2	$1.0(1).\mathsf{B}(1).0(1).1.0(0).\mathsf{B}(2).\underline{1.0(0).1}$
3	$1.0(2).\mathsf{B}(2).0(3).1.0(1).\mathsf{B}(4).\underline{1.0(1).1}$
4	$1.0(3).\mathsf{B}(3).0(5).1.0(2).\mathsf{B}(6).\underline{1.0(2).1}$
5	$1.0(4).\mathsf{B}(4).0(7).1.0(3).\mathsf{B}(8).\underline{1.0(3).1}$

Here is the pattern:

$$1.0(k-1).\mathsf{B}(k-1).0(2k-3).1.0(k-2).\mathsf{B}(2k-2).\underline{1.0(k-2).1}$$

whenever $k \geq 2$.

Universal k-Phoenix Numbers, $k \geq 2$, I.c

Theorem 11

For $k \geq 2$, the $(8k-7)$-digit number

$$1.0'.\mathsf{B}'.0''.0.0''.1.0''.\mathsf{B}'.\mathsf{B}'.\underline{1.0(k-2).1} \tag{2}$$

is k-Phoenix in any base b, where $\mathsf{B} = b - 1$ and

$$\mathsf{B}' = \mathsf{B}(k-1), \quad \mathsf{B}'' = \mathsf{B}(k-2), \quad 0' = 0(k-1), \quad 0'' = 0(k-2).$$

Therefore, k-Phoenix numbers exist in any base b, $k \geq 2$.

Proof.

		B'	B''	B	$0''$	1	$0''$	$0'$	B'	0	$0''$	1
										1	$0''$	1
(1)	$(1)'$	$(1)'$	$(1)''$									
		B'	B''	B	$0''$	1	$0''$	$0'$	B'	0	$0''$	1
	B'	B'	$0''$	1	$0''$	0	$0''$	B'	$0'$	1		
1	$0'$	B'	$0''$	0	$0''$	1	$0''$	B'	B'	1	$0''$	1

Base $b \geq 2$.

Universal k-Phoenix Numbers, $k \geq 2$, II.a

The following patterns are observed.

k	
2	1.0(1).B(1).0(1).1.0(0).[B(1).C.B(0).0(1).1.0(0)].B(2).$\underline{1.0(0).1}$
3	1.0(2).B(2).0(3).1.0(1).[B(3).C.B(1).0(3).1.0(1)].B(4).$\underline{1.0(1).1}$
4	1.0(3).B(3).0(5).1.0(2).[B(5).C.B(2).0(5).1.0(2)].B(6).$\underline{1.0(2).1}$
5	1.0(4).B(4).0(7).1.0(3).[B(7).C.B(3).0(7).1.0(3)].B(8).$\underline{1.0(3).1}$

Universal k-Phoenix Numbers, $k \geq 2$, II.b

Theorem 12

For any base $b \geq 2$ and $k \geq 2$, let $\mathsf{B} = b - 1$, $\mathsf{C} = b - 2$, *and let*

$$\mathsf{A}_0 = \mathsf{B}''.\mathsf{B}.\mathsf{B}''.\mathsf{C}.\mathsf{B}''.0''.0.0''.1.0''. \tag{3}$$

Then the number

$$1.0'.\mathsf{B}'.0''.0.0''.1.0''.[\mathsf{A}_0].\mathsf{B}'.\mathsf{B}'.\underline{1.0''.1} \tag{4}$$

is k-Phoenix in base b. Thus, there are infinitely many k-Phoenix numbers in base b.
Furthermore, if there are m occurrences of A_0, *then the number* (4) *has*

$$8k - 7 + 6(k-1)m$$

digits.

Universal k-Phoenix Numbers, $k \geq 2$, II.c

Theorem 13

For any base $b \geq 2$ and $k \geq 2$, let $\mathsf{B} = b - 1$, $\mathsf{C} = b - 2$, and let

$$\mathsf{A}_1 = \mathsf{B}'.\mathsf{B}'.0', \quad (5)$$

$$\mathsf{A}_2 = \mathsf{B}'.0'.0'. \quad (6)$$

Then the number

$$1.0'.[\![\mathsf{A}_2]\!].\mathsf{B}'.0''.0.0''.1.0''.[\![\mathsf{A}_1]\!].\mathsf{B}'.\mathsf{B}'.\underline{1.0''.1} \quad (7)$$

is k-Phoenix in base b. Thus, there are infinitely many k-Phoenix numbers in base b.
Furthermore, if there are n occurrences each of A_1 and A_2, then the number (7) has

$$8k - 7 + 6(k - 1)n$$

digits.

Universal k-Phoenix Numbers, $k \geq 2$, II.d

Theorem 14

For any base $b \geq 2$ and $k \geq 2$, let $\mathsf{B} = b - 1$, $\mathsf{C} = b - 2$, and let

$$\mathsf{A}_0 = \mathsf{B}''.\mathsf{B}.\mathsf{B}''.\mathsf{C}.\mathsf{B}''.0''.0.0''.1.0'', \tag{8}$$

$$\mathsf{A}_1 = \mathsf{B}'.\mathsf{B}'.0', \tag{9}$$

$$\mathsf{A}_2 = \mathsf{B}'.0'.0'. \tag{10}$$

Then the number

$$1.0'.[\![\mathsf{A}_2]\!].\mathsf{B}'.0''.0.0''.1.0''.[\mathsf{A}_0].[\![\mathsf{A}_1]\!].\mathsf{B}'.\mathsf{B}'.\underline{1.0''.1} \tag{11}$$

is k-Phoenix in base b. Thus, there are infinitely many k-Phoenix numbers in base b.
Furthermore, if there are m occurrences of A_0 and n occurrences each of A_1 and A_2, then the number (11) *has*

$$8k - 7 + 6(k-1)m + 6(k-1)n$$

digits.

2-Phoenix Numbers With Four Digits I.a

Consider the following pair of examples of 2-Phoenix numbers with four digits:

248	10.25.10.100
249	15.25.10.150

It's the second example that I worked on first.

2-Phoenix Numbers With Four Digits I.b

The number $15.25.10.150$ in base $b = 249$ suggests the pattern $x.x + y.y.xy$ in base $b = xy + y^2 - 1$. Here is the multiplication table.

		(9)	
		(0)	
		25	15
		150	10
	1	1	150
15	24	9	
15	25	10	150

Base $b = 249$.

		$(y-1)$	
		(0)	
		$x+y.$	x
		$xy.$	y
	$1.$	$1.$	xy
x	$x+y-1.$	$y-1.$	
x	$x+y.$	$y.$	xy

Base $b = xy + y^2 - 1$.

(12)

2-Phoenix Numbers With Four Digits I.c

The equations defining the multiplication (12) are

$$x^2y = (y-1)(xy+y^2-1)+(y-1), \tag{13a}$$

$$xy(x+y)+(y-1) = x(xy+y^2-1)+(x+y-1). \tag{13b}$$

Observe that the RHS of (13b) is $x(xy+y^2-1)+(x+y-1) = x^2y+xy^2-x+x+y-1 = x^2y+xy^2+y-1$ and this is the same as the LHS.
(13a) simplifies to

$$y^2+(x-1)y-(x^2+x) = 0. \tag{14}$$

Solving for y, we obtain

$$y = \frac{1}{2}\left(\sqrt{5x^2+2x+1}-x+1\right). \tag{15}$$

2-Phoenix Numbers With Four Digits I.d

Using the computer to generate some examples, we find the following.

b	$x.x + y.y.xy$
7	2.4.2.4
249	15.25.10.150
10984	104.169.65.6760
510951	714.1156.442.315588
23968945	4895.7921.3026.14812270
1125790992	33552.54289.20737.695767824

A081018 $a(n) = F_{2n-1} \cdot F_{2n}$, $n \geq 0$.

$$0, 2, 15, 104, 714, 4895, 33552, \ldots$$

A064170 $a(n) = F_{2n-1} \cdot F_{2n+1}$, $n \geq 1$.

$$2, 10, 65, 442, 3026, 20737, \ldots$$

A081068 $a(n) = F_{2n+1}^2$, $n \geq 0$.

$$1, 4, 25, 169, 1156, 7921, 54289, \ldots$$

A000045 Fibonacci numbers: $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

2-Phoenix Numbers With Four Digits II.a

Theorem 15

The number $x.x + y.y.xy$ *is 2-Phoenix in base* $b = xy + y^2 - 1$*, where* $x = F_{2n}F_{2n+1}$ *and* $y = F_{2n-1}F_{2n+1}$*,* $n \geq 1$*. Furthermore,* $x + y = F_{2n+1}^2$ *and* $b = F_{2n-1}F_{2n+1}^3 - 1$*.*

Proof.

It is now a straightforward application of Fibonacci identities to show that

$$\begin{aligned} x + y &= F_{2n-1}F_{2n+1} + F_{2n}F_{2n+1} \\ &= (F_{2n-1} + F_{2n})F_{2n+1} \\ &= F_{2n+1}^2. \end{aligned}$$

Furthermore,

$$\begin{aligned} b &= (F_{2n-1}F_{2n+1})^2 + (F_{2n-1}F_{2n+1})(F_{2n}F_{2n+1}) - 1 \\ &= F_{2n-1}^2F_{2n+1}^2 + F_{2n-1}F_{2n}F_{2n+1}^2 - 1 \\ &= F_{2n-1}(F_{2n-1} + F_{2n})F_{2n+1}^2 - 1 \\ &= F_{2n-1}F_{2n+1}^3 - 1 \end{aligned}$$

2-Phoenix Numbers With Four Digits II.b

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Theorem 16

The number

$$n(n^2+1).(n^2+1)^2.\underline{n(n^2+1).n^2(n^2+1)^2}$$

is 2-Phoenix in base $b = n(n^2+1)^3 - n$, $n \geq 1$. *See Table 1.*

Proof.

By multiplication.

Multiplication for Theorem 16.

		(n) $(n^2+1)^2.$ $n^2(n^2+1)^2.$	 $n(n^2+1)$ $n(n^2+1)$
 $n(n^2+1).$	$1.$ $n^4+2n^2.$	$n.$ n^3	$n^2(n^2+1)^2$
$n(n^2+1).$	$(n^2+1)^2.$	$n(n^2+1).$	$n^2(n^2+1)^2$

Base $b = n(n^2+1)^3 - n$.

2-Phoenix in Base $b = 2n^2 - 1$

An interesting observation is that $2.4.2.4$ occurs as the initial term of both sequences. Even more interesting is that $2.4.2.4$ occurs as the initial term of a third sequence!

Theorem 17 (2-Phoenix in Base $b = 2n^2 - 1$)

If $n \geq 2$, the number

$$2(n-1).2n(n-1).\underline{2(n^2-n-1).2n}$$

is 2-Phoenix in base $b = 2n^2 - 1$.

Proof.

By multiplication.

		(1)	
		$(2n-4)$	
		$2n^2-2n.$	$2n-2.$
		$2n.$	$2n^2-2n-2$
	$2n^2-4n+1.$	$2n-3.$	$2n$
$2n-2.$	$2n-1.$	$2n^2-4n+1.$	
$2n-2.$	$2n^2-2n.$	$2n^2-2n-2.$	$2n$

Theorem 17, base $b = 2n^2 - 1$.

Two More Interesting Examples of 2-Phoenix Numbers

		(n)	
		$8n^2+2.$	$2n$
		$4n^2.$	$8n^3+3n^2$
	$4n^2+2.$	$3n.$	$4n^2$
$2n.$	$4n^2.$	$8n^3$	
$2n.$	$8n^2+2.$	$8n^3+3n.$	$4n^2$

Base $b = 16n^3 + 2n.$

		$9n^2+2.$	$3n$
		$6n^2.$	$2n$
	(1)		
	$1.$	$3n.$	$6n^2$
$3n.$	$9n^2.$	$18n^3$	
$3n.$	$9n^2+2.$	$2n.$	$6n^2$

Base $b = 18n^3 + n.$

3-Phoenix numbers, 1

Theorem 18 (3-Phoenix in base $b = 14 + 12k$)
The number

$$3.4.\underline{13 + 11k.1 + 2k.11 + 9k}$$

is 3-Phoenix in base $b = 14 + 12k$, $k \geq 0$.

k	b	
0	14	3.4.13.1.11
1	26	3.4.24.3.20
2	38	3.4.35.5.29
3	50	3.4.46.7.38
4	62	3.4.57.9.47
5	74	3.4.68.11.56
6	86	3.4.79.13.65

3-Phoenix numbers, 2

Definition 19 (u-property)

A number base b has the u-property if given u, $1 < u < b$, there exists v such that $uv = rb$, $r = b - u$, or, equivalently, $uv = r.0$ and $u + r = 1.0$ in base b.

- The base b has the u-property if and only if $u(b + v) = b^2$ for some v. Neither u nor v are necessarily divisors of b. It is easily verified that neither u nor v can be $b - 1$, and so $2 \leq u, v \leq b - 2$.

- The sequence of bases b with the u-property seem to be the same as sequence A005279 in the *Online Encyclopedia of Integer Sequences*, namely, numbers having divisors d, e with $d < e < 2d$. A proof of equality would be nice!

3-Phoenix numbers, 3

Theorem 20 (3-Phoenix numbers with u-property)

The number $1.0.0.u.\underline{1.v.1}$ *is* 3*-Phoenix in base* b *if and only if* b *has the* u*-property.*

Proof.

By multiplication. □

Table: Proof of Theorem 20.

				u	0	0	1
					1	v	1
1.				u	0	0	1
2.		r	0	0	0	v	
3.		u	0	0	1		
	1	0	0	u	1	v	1

Base b has the u-property:

$$uv = r.0,\ r = b - u.$$

3-Phoenix numbers, 4

The following table is in base 12.

b	1.0.0. u.1. v.1. r
6	1.0.0. 4.1. 3.1. 2
10	1.0.0. 8.1. 6.1. 4
10	1.0.0. 9.1. 4.1. 3
13	1.0.0. 9.1. X.1. 6
16	1.0.0.10.1. 9.1. 6
18	1.0.0.14.1. 5.1. 4
20	1.0.0.14.1.10.1. 8
20	1.0.0.16.1. 8.1. 6
24	1.0.0.14.1.19.1.10
26	1.0.0.16.1.18.1.10
26	1.0.0.18.1.13.1. X
26	1.0.0.21.1. 6.1. 5
2E	1.0.0.21.1.12.1. X
30	1.0.0.20.1.16.1.10
30	1.0.0.23.1.10.1. 9

4-Phoenix numbers, 1

The only discernable pattern among 4-Phoenix numbers with six digits is $1.p.\underline{x.y.z.x}$.

b	N	N_b
7	29808	1.5.2.6.2.2
112	25496485731	1.50.3.100.26.3

		(2)	(4)	(1)	
		2.	2.	6.	2
				5.	1
			(1)		
		2.	2.	6.	2
1.	5.	0.	3.	3.	
1.	5.	2.	6.	2.	2

Base $b = 7$.

		(12)	(44)	(1)	
		3.	26.	100.	3
				50.	1
			(1)		
		3.	26.	100.	3
1.	50.	0.	73.	3.	
1.	50.	3.	100.	26.	3

Base $b = 112 = 2^4 \cdot 7$.

After a bit of searching for numbers with similar multiplication only one new number was found.

b	N	N_b
7	29808	1.5.2.6.2.2
112	25496485731	1.50.3.100.26.3
9286	127438368506940868182	1.7853.2.4560.1694.2

Vampire numbers

Definition 21 (Vampire number)

If a product contains exactly the same digits as its factors, expressed in base b, is called a vampire number in base b.

- Any k-Phoenix number is also a k-vampire number.
- Here's a table of vampire numbers with two factors in base 10 in which the factors are both less than 100.

(3)(51) = 153
(6)(21) = 126
(8)(86) = 688
(15)(93) = 1395
(21)(60) = 1260
(21)(87) = 1827
(27)(81) = 2187
(30)(51) = 1530
(35)(41) = 1435
(80)(86) = 6880

- From now on we exclude the case when one of the factors is zero modulo the base, since, for example, $(30)(51) = 1530$ is essentially the same as $(3)(51) = 153$.

Vampire numbers with three factors

Here's a table of all base 10 vampire numbers with all three factors less than 100.

(2)(79)(81) = 12798
(4)(72)(86) = 24768
(5) (9) (31) = 1395
(5)(49)(81) = 19845
(6)(31)(74) = 13764
(6)(73)(86) = 37668
(6)(84)(99) = 49896
(8)(21)(76) = 12768
(8)(31)(66) = 16368
(9)(28)(71) = 17892
(9)(31)(41) = 11439
(21)(61)(95) = 121695
(21)(74)(82) = 127428
(29)(75)(91) = 197925
(46)(72)(89) = 294768
(56)(87)(94) = 457968
(57)(75)(93) = 397575
(69)(74)(94) = 479964

Vampire numbers in base 12 with three factors

Here are some examples.

(4)(87)(E2) = 27E48
(5) (7) (X2) = 257X
(6)(51)(6E) = 156E6
(7)(34)(81) = 13874
(8)(37)(81) = 17388
(X) (E) (75) = 57EX
(E)(63)(76) = 36E76
(21)(89)(99) = 129899
(25)(X6)(E1) = 1E52X6
(31)(71)(X6) = 1713X6
(49)(65)(81) = 186459
(5E)(61)(X2) = 265E1X
(66)(E5)(E7) = 5E76E6
(74)(81)(E2) = 471E28
(81)(84)(X5) = 4X5818
(84)(9E)(X5) = 5E89X4

Vampire number patterns 1/3

Theorem 22

If $d, e \geq 2$*, then*

$$(d+1)(e.[0].1) = 1.e.[0].d+1$$

in base $b = de$*.*

Theorem 23

If $n \geq 3$*, then*

$$(2n-1)(n-2.n.1) = n-2.1.n.2n-1$$

in base $b = 2n$*.*

Theorem 24

If $n \geq 2$*, then*

$$(n.1)(n.0.1) = 1.0.n.n.1$$

in base $b = n^2$*.*

Vampire number patterns 2/3

Theorem 25

If $n \geq 2$, then

$$(1.n+3)(n^2+n-1.1) = 1.1.n^2+n-1.n+3$$

in base $b = n^2+2n-1$.

Theorem 26

If $n \geq 2$, then

$$(n.n)(2n-1.1) = 1.n.2n-1.n$$

in base $b = 2n^2-2n+1$.

Vampire number patterns 3/3

Theorem 27

If $n \geq 2$, *then*

$$(1.0.n{+}2)(n^2{+}n{-}1.n^2.1) = 1.0.1.n^2{+}n{-}1.n{+}2.n^2 \quad (b = n^2{+}n).$$

Theorem 28

If $n \geq 2$, *then*

$$(n.0.n)(n + 2.1.n + 1) = 1.n.n + 2.n.n + 1.0 \quad (b = n^2 + n).$$

Theorem 29

If $n \geq 2$, *then*

$$(n{+}1.0.1)(n{+}1.0.n^2{+}n{-}1) = 1.n{+}1.n{+}1.0.0.n^2{+}n{-}1 \quad (b = n^2{+}n).$$

Universal Vampire numbers 1/8

Definition 30

A universal vampire number is a digit pattern that holds for all bases $b \geq b_{\min}$, where $b_{\min}$ is usually 3 or 4. The smallest set of digits that contains all the universal Phoenix numbers consists of only a subset of the digits

$$U = \{0, 1, 2, \mathsf{D} = b - 3, \mathsf{C} = b - 2, \mathsf{B} = b - 1\}$$

in any base $b \geq b_{\min}$.

• The basic idea is to check that the full multiplication table consists of only digits in U for a sufficiently large set of bases b. The pattern will then hold for all bases b, $b \geq b_{\min}$. Spot checking once in awhile hasn't failed ... yet.

Example 31

It is easy to show that

$$\mathsf{B}(\mathsf{DB}11) = \mathsf{D}11\mathsf{BB}$$

for all bases $b \geq 4$.

Universal Vampire numbers 2/8

Theorem 32

B(DB11) = D11BB *for all bases* $b \geq 4$.

Proof. Since $\mathsf{B}^2 = \mathsf{C}.1$ and $\mathsf{B} \cdot \mathsf{D} + \mathsf{C} = \mathsf{D}.1$, we have

	(C)			
	D	B	1	1
				B
D	1	1	B	B

Universal Vampire numbers 3/8

Theorem 33

The following pattern is universal 1-vampire pattern.

$$(\mathsf{B})(\mathsf{D.B.}[\mathsf{B}].[0].1.[0].1) = \mathsf{D}.1.[\mathsf{B}].1.[0].\mathsf{B}.[0].\mathsf{B}.$$

Universal Vampire numbers 4/7

Theorem 34

The following pattern is a universal 1-vampire pattern for all bases $b \geq 4$.

$$(\mathsf{B})(\mathsf{D.B.}[\mathsf{B}].[0].1.[0].1) = \mathsf{D}.1.[\mathsf{B}].1.[0].\mathsf{B}.[0].\mathsf{B}.$$

Universal Vampire numbers 5/8

Theorem 35

The following patterns are universal 1-vampire patterns for all bases $b \geq 4$.

1.	(B)(D.⟦B0C⟧.B.1.⟦1⟧.1)=D.1.1.⟦C01⟧.B.⟦B⟧.B
2.	(B)(D.⟦CB0⟧.B.1.⟦1⟧.1)=D.1.⟦01C⟧.1.B.⟦B⟧.B
3.	(B)(D.B.⟦1⟧.1.⟦0CB⟧.1)=D.1.1.⟦B⟧.B.⟦C01⟧.B
4.	(B)(D.⟦C⟧.B.⟦10B⟧.1.1)=D.1.⟦0⟧.1.⟦BC1⟧.B.B
5.	(B)(D.⟦C⟧.B.⟦0B1⟧.1.1)=D.1.⟦0⟧.1.⟦C1B⟧.B.B
6.	(B)(D.⟦C⟧.B.1.⟦10B⟧.1)=D.1.⟦0⟧.1.B.⟦BC1⟧.B

Universal Vampire numbers 6/8

Theorem 36

The following patterns are universal 1-vampire patterns for all bases $b \geq 4$.

1.	(B)(DB[0CB1]11)=D11[C01B]BB
2.	(B)(D[CB10]B11)=D1[01BC]1BB
3.	(B)(DB1[10CB]1)=D11B[BC01]B
4.	(B)(DB[10CB]11)=D11[BC01]BB

Universal Vampire numbers 7/8

The table below shows all possible universal vampire with three factors and all factors having at most four digits.

1.	(B1)(CD12)(BBCB)=DB1B1BB2CC
2.	(B1)(CDB2)(BBBB)=DBBBB1BB2C
3.	(C1B)(BB01)(BBCB)=C0BBBBB1CB1
4.	(CB1)(B101)(BCCB)=C01CBC11BBB
5.	(B01)(B0B)(BCB)=C0BBBBB01
6.	(CDB1)(B1B1)(BC01)=DB1B01BB1CC1

Universal Vampire numbers 8/8

Note the following calculations. The intermediate products are not vampire numbers, but the final result is.

$$(B01)(B0B) = C1B00B$$
$$(BCB)(C1B00B) = C0BBBBB01$$

$$(B01)(BCB) = CB10CB$$
$$(B0B)(CB10CB) = C0BBBBB01$$

$$(B0B)(BCB) = CBB001$$
$$(B0B)(CBB001) = C0BBBBB01$$

Thank you for the invitation! It was a pleasure presenting these results.