**The Battleship Game**

This game is played in a 4x4 grid. Player 1 anchors a three-cell-long battleship (horizontally or vertically) somewhere on the grid. There are sixteen possible positions, eight horizontal and eight vertical. The eight of the sixteen that contain a corner cell are “Class A” positions. The eight that contain a non-corner edge cell are “Class B” positions.

Player 2 does not know where Player 1's battleship is anchored. He drops bombs sequentially on cells of the grid, and after each bomb he is told whether he has hit the battleship. His goal is to sink the battleship (by bombing all three cells) in as few moves as possible.

Player 1 will make his choice of a position for his battleship by a random device. Since the eight Class A positions are all symmetric images of one another under symmetries of the square, they must have the same probabilities. Similarly for Class B. Let a denote the probability of choosing Class A and b denote the probability of choosing Class B. Note that a + b = 1; also, that the probability of choosing any particular Class A position is a/8 and that of choosing any particular Class B position is b/8. We will eventually determine values for a (and therefore b).

The cells can be partitioned into three groups: four corner cells, four center cells, and eight (non-corner) edge cells. Each corner cell belongs to two battleship positions, both of Class A. Each edge cell belongs to three battleship positions, two of Class A and one of Class B. Each center cell belongs to four battleship positions, all of Class B. Note that it is pointless to bomb a corner cell, as more (or at least as much) information is always gained by bombing an edge cell instead.

Player 2's strike sequence can be divided into two episodes: search episode and destroy episode. The search episode continues until he makes a hit; thereafter he is in destroy episode, since he then has significant information about the possible positions of the battleship.

We will denote by E an attack on an edge cell and by C an attack on a center cell. Following are possible sequences for the search episode. The E's in any sequence are assumed to be on distinct sides, chosen in random order. Unless otherwise specified the E on a side is either edge cell, chosen at random.

1) EEEEC. The E's on top and bottom may be on a pair of cells that are reflectively symmetric or centrally symmetric; likewise the E's on right and left. It is best to make the last choice of E so that one pair is reflectively symmetric and the other is centrally symmetric. The C is …

If the first E is a hit, then Player 2 moves into destroy mode. He strikes the adjacent edge cell. If that is a miss, the battleship is known to be perpendicular to the edge and can be sunk with two more bombs. If that is a hit, the battleship is along the edge in one of 2 positions. Player 2 may guess correctly and sink it in one more bomb, or guess incorrectly and sink it in two more. [This is Strategy 2a; must include 2b; however, it is superceded.]

He will in this case have discovered one Class A position in 3 moves, a second in 4 moves, and a Class B position in 4 moves.

If his first E is a miss but the second is a hit, the situation proceeds similarly. He will have discovered a third Class A position in 4 moves, a fourth in 5 moves, and a second Class B position in 6 moves. These numbers are respectively one more than the previous case since one previous bomb was a miss.

We can proceed similarly for the cases that the third E is the first hit and the fourth E is the first hit.

If no E's are hits, he has eliminated all 8 Class A positions and 4 of the 8 Class B positions. It can be arranged that the two horizontal B positions remaining are in different rows and the two vertical B lpositions remaining are in the same row, or vice versa. Player 2’s next move, C, should be a cell overlapping 3 B positions. If it’s a miss, he knows exactly where the ship is and finishes it in 3 more. Otherwise, he hits the adjacent center cell overlapping two positions. If it is a hit, he knows the battleship's line (row or column) and destroys it in one more move if lucky or two more if not. If it is not a hit, he still knows the battleship's position and destroys it in 3 more bombs. Thus (having “wasted” 4 bombs fruitlessly searching on edges) he gets the ship in B positions in 7 or 8 or 8 or 8 moves.

Class A: 3 4 4 5 5 6 6 7 so A = 5

Class B: 4 5 6 7 7 8 8 8 so B = 6 5/8

The expected number of strikes to sink the battleship is thus M = 5 a + (6 5/8) b.

(Note this option is highly desirable when it is most likely Player 1 has chosen a Class A position.)

2) CEEEE. The C knocks out four Class B positions, one touching each edge. The E's should be chosen to avoid those edge cells that have been eliminated.

If the C is a hit, he then attacks one rectilinearly adjacent cell. If it is a hit, he knows the battleship's line and destroys it in 2 or 3 more moves. If it is a miss, he still know the battleship's line, though it has taken one more move, and he can again destroys it in 2 or 3 more moves.

If C is a miss, he begins sequentially on the E's as above [This is Strategy 3a; must include 3b; but again this strategy is dominated and need not be considered.]

Class A: 4 5 5 6 6 7 7 8 so A = 6

Class B: 3 4 4 5 5 6 7 8 so B = 5 1/4

The expected number of strikes to sink the battleship is thus M = 6 a + (5 1/4) b.

3) CCEEEE. The second C should be diagonally adjacent to the first. Now it doesn't matter which edge cell on an edge he hits, since all Class B positions have been eliminated. He proceeds through the E's as above.

Class A: 5 6 6 7 7 8 8 9 so A = 7

Class B: 3 4 4 5 4 5 5 6 so B = 4 ½

The expected number of strikes to sink the battleship is thus M = 7 a + (4 1/2) b.

(Note this option is highly desirable when it is most likely Player 1 has chosen a Class B position.)

There are many more permutations of C’s and E’s, but all others are domianted by these three. The three analyzed above supercede the others, so only those three are necessary. Now, we should graph all three lines of the form M = A a + B b, where M denotes the expected number of moves for Player 2 to win, as a function of a (and b). We look for the one that is lowest for any a. Then Player 1 looks for the highest of all such. It occurs when a = 11/19. Thus Player 1's optimal strategy is to select a Class A position at random with probability 11/19 (about 57.9%) and a Class B position at random with probability 8/19 (about 42.1%). Then Player 2 will be indifferent between his Strategies 1 and 2.

However, Player 2 must also think defensively. Since he is indifferent when Player 1 is playing optimally, he can mix Strategies 1 and 2 in any combination, and must do so in such a way that Player 1 has no incentive to abandon his mixed strategy for that of always choosing Class A or always Class B. We therefore graph two more lines, using the same numbers as the lines for Strategies 1 and 2, but putting the left numbers (a = 0, b = 1) at the ends of one line and the right numbers (b = 0, a = 1) at the ends of the other line. They cross at 13/19, which means Player 2's optimal strategy is to choose Strategy 1 with probability 13/19 (about 78.4%) and Strategy 2 with probability 6/19 (about 31.6%).

When both players are playing optimally, the expected length of the game is M = 108/19 (about 5.68).