Summation Methods on Divergent Series

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- Infinite series are incredibly useful tools in many areas of both pure and applied mathematics, as well as the sciences, especially physics.
- For example, Taylor series. We use Taylor series to approximate functions to any desired degree of accuracy which is computationally efficient if we can have an error so small its negligible.

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- However, the period of a pendulum is actually an elliptic integral which has no closed form solution and the above expression is actually just the first term in the Taylor series expansion, which for small amplitude oscillations is accurate "enough" (more terms are used for higher amplitudes or more accuracy).

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- often used in electrical engineering

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- Summation methods to the rescue!

- I will discuss two summation methods in particular, Cesaro summation and Abel summation.

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- In order to relate the two, there are some terms used to describe the relationships between summation methods.

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- Two summation methods are said to be consistent if, when both assign a value to a series, they assign the same value.
- A summation method X is said to be stronger than another summation method Y if X sums all the series that Y does, plus more. (That is, the series summable by Y are a subset of those summable by X).
- Abel summation and Cesaro summation are consistent methods, but Abel summation is a stronger method.

- Cesaro Summation:

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- It is based on the Cesaro mean, which is really just the same idea applied to any sequence, not just a sequence of partial sums.
- It is defined as the limit of the arithmetic mean of the partial sums:

- Let $\{a_n\}$ be any real sequence, and let $s_n = \sum_{k=1}^{n} a_n$ be the nth partial sum.
- Then $(C,1)s_n = \sum_{k=1}^{n} \frac{s_k}{k}$ is the nth Cesaro sum and we say
- $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{s_k}{k} = L \quad (C,1),$ provided the limit exists.

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- If a series converges, it gets infinitely close to its limit. So for any arbitrarily small value, there is some finite corresponding index value, such that every partial sum past that index value is within that arbitrarily small value of the limit.
- Lets call that index N.

- Now, $$\lim_{n\to\infty}\frac{s_1+s_2+\ldots+s_n}{n}=\lim_{n\to\infty}\frac{s_1+s_2+\ldots+s_N}{n}+\lim_{n\to\infty}\frac{s_{N+1}+s_{N+2}+\ldots+s_n}{n}$$

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- Now we just need to consider the second term:

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- But since every partial sum past N is arbitrarily close to the limit, the average of all these sums will be within the same range of the limit!

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- If $a < b < c$, then $3a < a + b + c < 3c \rightarrow 3a < \frac{a+b+c}{3} < 3c$
- In general, if $x_1 < x_2 < \ldots < x_n$, then $x_1 < \frac{x_1 + x_2 + \ldots + x_n}{n} < x_n$
- …and so on. (This does not address the infinite case and is not meant as a proof, just intended to be an intuitive explanation).
- Proof can be found in Mathematical Analysis by Tom Apostol, p.206

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- Let's examine Grandi's series: $1-1+1-1+1-\ldots$
- Now let's consider the partial sums of the series:

$$s_1 = 1,\ \ s_2 = 0,\ \ s_1 = 1,\ \ s_4 = 0\ldots$$

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- All even terms are 1/2 and the odd terms are converging to 1/2, so we say 1-1+1-1+1-… = 1/2 (C,1).

- We can now go further and define the second order Cesaro sum as the limit of the arithmetic mean of the partial (C,1) sums of the series:

- $(C,2)s_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(C,1)s_k}{k}$, provided the limit exists.

- More generally, for any positive integer m we can define the mth order Cesaro sum as:

$$(C,m)s_n = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{(C,m-1)s_k}{k}$$

- provided the limit exists.

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- The partial sums are:

$$s_1 = 1,\ \ s_2 = -1,\ \ s_3 = 2,\ \ s_4 = -2,...$$

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- Lets apply Cesaro summation to the Cesaro sum!
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- The second order Cesaro sums are:

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$$(C,2)s_5 = \frac{1+0+\frac{2}{3}+0+\frac{3}{5}}{5} = \frac{34}{75}$$

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- Considering this is an averaging method, this should make sense since the first order Cesaro terms oscillate between 0 and the terms which converge to 1/2.

- One area that Cesaro summation is extremely useful in is Fourier anylsis.

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- One area that Cesaro summation is extremely useful in is Fourier anylsis.
- Fourier series are the result of the fact that any periodic function, no matter how complicated, can be expressed as a (possibly infinite) linear combination of sine and cosine functions.
- However, sometimes the Fourier series “supposedly” modeling a particular function diverges.
- What good is that?

- Fejer's Theorem: If f is a continuous function with period 2π then the first order Cesaro sum of the Fourier series of f converges to f.

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- Even if the Fourier series representation of f diverges, the Cesaro sum of the series will converge to f!

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- for positive integers m.
- Euler actually thought about this before Niels Abel was even born, but his thoughts about this are very similar to what is now referred to as Abel summation.

- We define the Abel sum of some sequence $\{a_n\}$as:

$$\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n$$

- As x approaches 1 from below (taking a limit to the end of the interval of convergence) the Abel sum becomes the original sum.

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$$\lim_{x\to 1^-} \sum_{n=0}^{\infty} x^n = \lim_{x\to 1^-} \frac{1}{1-x}$$

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- As x approaches 1, we assign the same sum for Grandi's series as Cesaro summation does:

$$1-1+1-1+1-\ldots = \frac{1}{1+1} = \frac{1}{2}$$

- Now Euler’s cleverness:

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- Let's multiply both the alternating geometric series and its sum by x to obtain

$$x - x^2 + x^3 - x^4 + \ldots = \frac{x}{1+x}$$

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- and take the derivative of both sides with respect to x so that

$$1 - 2x + 3x^2 - 4x^3 + ... = \frac{1}{(1+x)^2}$$

- Again taking the limit as x approaches one (from above)

$$\lim_{x\to1^+} 1-2x+3x^2-4x^3+\ldots=\lim_{x\to1^+}\frac{1}{(1+x)^2}$$

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$$\lim_{x\to 1^+} 1-2x+3x^2-4x^3+\ldots = \lim_{x\to 1^+} \frac{1}{(1+x)^2}$$

- we get

$$1-2+3-4+\ldots = \frac{1}{4}$$

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- Again taking the limit as x approaches one (from above)

$$\lim_{x \to 1^{+}} 1 - 2x + 3x^{2} - 4x^{3} + \ldots = \lim_{x \to 1^{+}} \frac{1}{(1+x)^{2}}$$

- we get

$$1 - 2 + 3 - 4 + \ldots = \frac{1}{4}$$

- Which again, is consistent with the Cesaro sum.
-

- Applying this process again (multiplying by x and differentiating), we have:

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- and as x approaches 1 (from above), we get:

$$1-2^2+3^2-4^2+\ldots=0$$

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$$1-2^3+3^3-4^3+\ldots=-\frac{1}{8}$$

- Further iteration of the same process results in the following series:

$$1 - 2^3 + 3^3 - 4^3 + \ldots = -\frac{1}{8}$$

$$1 - 2^4 + 3^4 - 4^4 + \ldots = 0$$

- These are just two of many summation methods:
- Hausdorff transformations
- Hölder summation
- Hutton's method
- Ingham summability
- Lambert summability
- Le roy summation
- Mittag-Leffler summation
- Ramanujan summation
- Riemann summability
- Riesz means
- Vallée-Poussin summability

- References:

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