A Surprising Connection between Two Proofs of the Infinitude of Primes

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Euclid of Alexandria, 300BC

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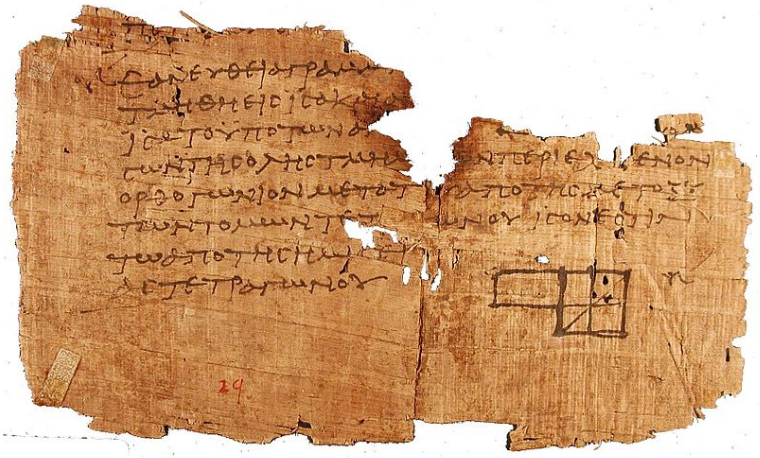
- deduced, in *The Elements*, the principles of what is now called Euclidean geometry from a small set of axioms.
- other non-Euclidean geometries emerged in the late 19th century.



one of the oldest surviving fragments of *The Elements*, 100AD

Euclid, depicted in Rafael's *School of Athens* (1510)

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Definition

- A prime number is a natural number greater than 1 that has no positive divisors other than 1 and itself.
- A composite number is a natural number greater than 1 that is not prime.

Note the number 1 is neither a prime nor composite. It is generally referred to as a *unit*.

In Book 9 of *The Elements*, Euclid established the following.

Main Theorem

There exists an infinite number of primes.

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- and more.

Euclid's Proof (300BC)

First we recall the all-important ...

Fundamental Theorem of Arithmetic

For all $n \in \mathbb{Z}$ such that $n > 1$, n can be represented uniquely as the product of primes.

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Definition

Let $a, m \in \mathbb{Z}$. An arithmetic sequence is a set of integers of the form

$$a + m\mathbb{Z} = \{a + mn : n \in \mathbb{Z}\}$$

For example, the set $\{..., -11, -4, 3, 10, 17, 24, ...\}$ is an arithmetic progression, where $a = 7$ and $m = 3$.

Lemma

For all integers m not equal to -1 *or* 1,

$$m\mathbb{Z} + 1 \subseteq \mathbb{Z} \backslash (m\mathbb{Z}).$$

I.e., one more than a multiple of m is not a multiple of m.

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Proof.

Let $mk + 1 \in m\mathbb{Z} + 1$ for some $k \in \mathbb{Z}$. Suppose by way of contradiction that $mk + 1$ is a multiple of m. Then there exists $n \in \mathbb{Z}$ such that $mk + 1 = mn$. Thus $1 = m(n - k)$ and m divides 1. So m must be either -1 or 1, which is contradiction. We conclude $m\mathbb{Z} + 1 \subseteq \mathbb{Z}\backslash(m\mathbb{Z})$. □

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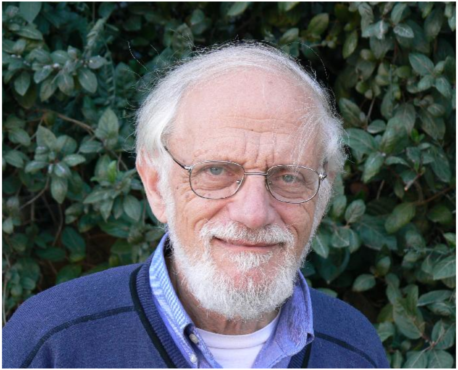
- Since for any finite list F of primes there is a prime not on our list, we conclude the set P of primes is infinite.

Hillel Furstenberg



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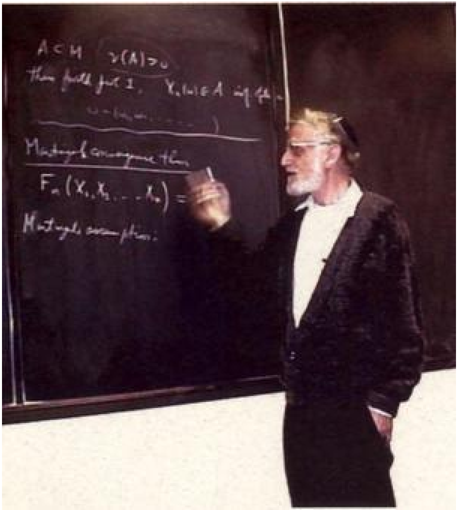
- currently at Hebrew University of Jerusalem, works in differential geometry and ergodic theory



The Evenly-Spaced Integer Topology on $\mathbb{Z}$

Definition

The evenly-spaced integer topology on $\mathbb{Z}$ consists of the following collection of open sets:

$$\{U \subseteq \mathbb{Z} : a\mathbb{Z} + b \subseteq U \text{ for some } a, b \in \mathbb{Z}\}.$$

In other words, a non-empty set of integers is open in this space if and only if it contains an arithmetic sequence.

This amazing thing about this topology is that it actually is a topology! To help see why, let's consider this question:

Question

What can we say about the intersection of finitely many arithmetic sequences? That is, what are the possibilities for

$$\bigcap_{i=1}^{n}(a_i + m_i\mathbb{Z}),$$

where $a_1, \ldots, a_n$ and $m_1, \ldots, m_n$ are integers?

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- It follows from the previous Lemma that the finite intersection of open sets is open in the Evenly-Spaced Integer Topology.
- Other conditions for a topology are also satisfied.

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1. an arithmetic sequence is both open and closed (clopen) Why?
2. a finite set is not open (unless it is empty) as it cannot contain an infinite arithmetic sequence.

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- This is a contradiction since finite sets cannot be open.

Mercer's Variation

In 2009, Mercer "unpackaged" the topology in Furstenberg's proof to uncover the underlying number theory. We give Mercer's proof, also published in the *Monthly*.

Lemma

If $m \geq 2$, then

$$\mathbb{Z} \backslash (m\mathbb{Z}) = (1 + m\mathbb{Z}) \cup \ldots \cup ((m-1) + m\mathbb{Z})$$

I.e., $\mathbb{Z} \backslash (m\mathbb{Z})$ is a finite union of arithmetic sequences.

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- it follows that

$$\{-1, 1\} = \mathbb{Z} \backslash (p_1 \mathbb{Z}) \cap \mathbb{Z} \backslash (p_2 \mathbb{Z}) \cap \cdots \cap \mathbb{Z} \backslash (p_n \mathbb{Z}).$$

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- so $\{-1, 1\}$ is then a finite intersection of finite unions of arithmetic sequences.

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- this is a contradiction, showing that the primes are infinite.

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- Observe that in Mercer's variation on Furstenberg's proof, the key idea is to show that A is infinite, contradicting that $A = \{-1, 1\}$. (Thus there must be an infinitude of primes).

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- Since the above holds for any $m \in \mathbb{Z}$, we see that A is infinite.

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- But $p_1 p_2 \cdots p_n + 1 > 1$ and $A = \{-1, 1\}$. This is a contradiction.

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- Both observations contradict that $A = \{-1, 1\}$.

N. A. Carlson, *A Connection between Furstenberg's and Euclid's proofs of the Infinitude of Primes*, Amer. Math. Monthly **121** (2014), 444.

Thank you!