CMC3 44th Annual Conference, Monterey, CA
December 9 – 10, 2016

# Understanding Black Holes With Elementary Calculus

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Problems adapted from: Taylor, E.F., and Wheeler, J.A. (2000) "Exploring Black Holes", Addison Wesley Longman, New York. ISBN: 0-201-38423-X

1. A geosynchronous circular orbit has a radius of about 4.22 x $10^7$ meters. Such an orbit has a period of one day, so it can be used to station a satellite above a particular region of the Earth. Suppose one clock is on the Earth's surface at $r_1$ = 6.38 x $10^6$ meters, and a second identical clock is in geosynchronous orbit directly above the first clock at $r_2$ = 4.22 x $10^7$ meters. According to the first clock, how long will it take before the two clocks go out of synchronization by one microsecond ($10^{-6}$ sec.)? Ignore the time dilation effect of special relativity.

The physics equation you need is: $\Delta\tau = \sqrt{1-\frac{2M}{r}}\ \Delta t$, where M = 4.4 x $10^{-3}$ is the mass of the Earth in meters, Δt is the time interval according to a clock very far away, and Δτ is the time interval with respect to a clock at distance r. Since M/r will be a small quantity, you will also want to use the following math results:

$$\sqrt{1-\frac{2M}{r}} \approx 1-\frac{M}{r},\quad \frac{1}{\sqrt{1-\frac{2M}{r}}} \approx 1+\frac{M}{r},\ \text{and } (1+\delta)^{-1} \approx 1-\delta.$$

2. Consider a black hole of mass M = 10 km. There are two stationary observers, one at r = 21 km, and the other at r = 22 km. The higher observer slowly lowers a tape measure down to the observer at r = 21 km. What will be the reading of the tape measure when the lower end of it reaches the lower observer?

The physics equation you need is: $d\sigma = \frac{dr}{\sqrt{1-\frac{2M}{r}}}$. After integrating both sides of this equation from r = 21 to r = 22, the length Δσ on the left side will be the desired length of the tape measure.

3. Suppose an observer in a small unpowered spaceship is released very far away from a black hole with M = 15km. The observer has a clock inside, which will read what we call the proper time $\Delta\tau$ for the trip into the black hole. The observer looks out the window of the spaceship and determines that she has just passed through the event horizon at r = 2M. How long does she have, according to her own clock, before she reaches r = 0 and the big crunch?

The physics equations you need are:

$$\left(1-\frac{2M}{r}\right)\frac{dt}{a\tau}=1, \qquad (1)$$

and

$$\frac{dr}{dt}=-\left(1-\frac{2M}{r}\right)\sqrt{\frac{2M}{r}}. \qquad (2)$$

The math equation you need is the chain rule:

$$\frac{dr}{d\tau}=\frac{dr}{dt}\frac{dt}{d\tau}. \qquad (3)$$

**SOLUTIONS**

**Solution to #1**: The physics equation applies to each clock:

$\Delta\tau_1=\sqrt{1-\frac{2M}{r_1}}\,\Delta t$, and $\Delta\tau_2=\sqrt{1-\frac{2M}{r_2}}$. Eliminate $\Delta t$ between these two equations, with the result:

$$\Delta\tau_1=\frac{\sqrt{1-\frac{2M}{r_1}}}{\sqrt{1-\frac{2M}{r_2}}}\Delta\tau_2\approx\left(1-\frac{M}{r_1}\right)\left(1+\frac{M}{r_2}\right)\Delta\tau_2$$

Now let $\Delta\tau_2=\Delta\tau_1+\delta$, and multiply out the right side above, keeping only terms up to first order in M/r. Also assume that $\delta \ll \Delta\tau_1$. The result is:

$\Delta\tau_1\approx\left(1-M\frac{(r_2-r_1)}{r_1r_2}\right)(\Delta\tau_1+\delta)$. We can rewrite this as:

$$\left(1+\frac{\delta}{\Delta\tau_1}\right)^{-1} \approx 1-\frac{\delta}{\Delta\tau_1} = 1-\ M\,\frac{(r_2-r_1)}{r_1 r_2}$$, which reduces to:

$\delta = 5.85 \times 10^{-10}\,\Delta\tau_1$, where we have put in the numerical values of $r_1$, $r_2$, and M. With $\delta = 10^{-6}$ second, we get $\Delta\tau_1 = 1703$ sec., about one-half hour. Note that in a microsecond, a light signal can travel 300 meters, so this error is enough to severely disrupt the global GPS system.

**Solution to #2**: Using an online integrator, or your calculator, obtain the result:

$$\Delta\sigma = \int_{21}^{22} \frac{dr}{\sqrt{1-\frac{20}{r}}} \approx 3.84$$ km. Note that the tape measure is apparently stretched out due to the radial curvature of space near the black hole. According to the far away observer, the separation between the other two observers is 22 – 21 = 1km. The integral is elementary, so I believe students could use the substitution: $u = \sqrt{1-\frac{20}{r}}$, followed by a partial fractions expansion to perform the integral.

**Solution to #3**: Solve equation (1) in the problem for dt/dτ:

$$\frac{dt}{d\tau} = \frac{1}{\left(1-\frac{2M}{r}\right)}.$$ Then use this and equation (2) to substitute for dr/dt and dt/dτ in equation (3), with the result: $\frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}}$. Rearrange this to obtain:

$$d\tau = -\sqrt{\frac{r}{2M}}\,dr.$$ We can integrate both sides from 2M to zero and obtain:

$$\tau = -\int_{2M}^{0} \sqrt{\frac{r}{2M}}\,dr = \frac{4}{3}M.$$ This is a simple power law integral.

With M = 15000 meters, we have τ = 20000 meters. The conversion factor between meters and seconds is the invariant speed of light, $c = 3 \times 10^8$ m/s. So 20000 meters is equal to $(20000)/(3 \times 10^8) = 6.67 \times 10^{-5}$ seconds.

The observer does not have enough time to suffer or contemplate her fate any longer after she reaches the event horizon. Before reaching the event horizon, there is still a chance that she might escape the clutches of the black hole.