Derivation of the Energy Equation

For simplicity, let us assume that a mass is falling radially into a black hole, although the same derivation would hold also if a tangential component of motion were present. Now suppose the falling mass emits thee pulses of light in very quick succession, so we have three events occurring at three fixed values of the radial coordinate r, with values r_0 , r_1 and r_2 respectively. Let us also take the times of the first and last events to be fixed at the values 0 and T. These times are with respect to an observer far away from the black hole.

In relativity, we use the principle that the proper time for a freely falling mass between two fixed endpoints is maximized. Proper time is the time interval recorded by a clock being carried with the mass as it falls. So any motion that deviates from free-fall motion between the same two fixed points would record a smaller elapsed proper time.

So the mass will arrive at the middle event (the second pulse of light) in a time t that gives maximal proper time. Recall that the Schwarzshild metric gives us an expression for elapsed proper time between two given events.

For the first two pulses, separated by fixed space interval dr_1, let the time interval be t. Recall that this is the time interval recorded by a far-away observer. Then the metric yields the following equation for the proper time interval between the first two pulses:

$$\tau_1^2 = \left(1 - \frac{2M}{r_0}\right) t^2 + (space\ terms) \qquad (1)$$

Note that I have used r_0 in the first term on the right. I justify this by our assumption that τ and t are very small intervals, so the first two pulses are located very close to r_0. Alternatively, you could use some appropriate average value of r between the first two pulses. This will not affect the result below, because we are regarding the proper time as a function of t only, and the spatial locations of the events are fixed.

The far-away time between the second and third pulses is (T – t), and we can use the metric to give an expression for the proper time between the second and third pulses:

$$\tau_2^2 = \left(1 - \frac{2M}{r_1}\right) (T - t)^2 + (space\ terms) \qquad (2)$$

Now let us take derivatives of equations (1) and (2) with respect to t, with the result:

$$\frac{d\tau_1}{dt} = \frac{\left(1-\frac{2M}{r_0}\right)t}{\tau_1}, \qquad \text{and} \qquad \frac{d\tau_2}{dt} = -\frac{\left(1-\frac{2M}{r_1}\right)(T-t)}{\tau_2}.$$

The total proper time between the first and last events is $\tau = \tau_1 + \tau_2$, so we need:

$\frac{d\tau_1}{dt} + \frac{d\tau_2}{dt} = 0$, which gives us the equation:

$$\left(1 - \frac{2M}{r_0}\right)\frac{t}{\tau_1} = \left(1 - \frac{2M}{r_1}\right)\frac{(T-t)}{\tau_2}.$$

A change of notation will make the significance of this equation more clear. Replace t and τ with dt and $d\tau$, and let $(T - t) = dt_2$. Then we can put the equation into the form:

$\left\{\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}\right\}_1 = \left\{\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}\right\}_2$, where I have also re-labeled the r coordinates. We see that the quantity:

$$\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$$

is a constant of the motion. To find the physical significance of the constant, let the r coordinate go to infinity, where special relativity applies. In special relativity, the value of $(dt/d\tau)$ is energy per unit mass: E/m. So we have:

$$\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = \frac{E}{m}. \qquad (3)$$

If the particle is at rest at infinity, then E /m = 1, so in this case, the right side of equation (3) would be one.