The Limit of Humanly Knowable Mathematical Truth

Gödel's Incompleteness Theorems, and Artificial Intelligence

Tim Melvin

Santa Rosa Junior College

December 12, 2015

Another title for this talk could be...

An Argument Against

an Argument Against Artificial Intelligence Using Gödel's Incompleteness Theorems

Another title for this talk could be...

An Argument Against

an Argument Against Artificial Intelligence Using Gödel's Incompleteness Theorems

Using Gödel's Incompleteness Theorems

Another title for this talk could be...

An Argument Against

an Argument Against Artificial Intelligence Using Gödel's Incompleteness Theorems

Using Gödel's Incompleteness Theorems

Axiomitization of Mathematics

Why?

Axiomitization of Mathematics

Why?

Guido Grandi's "Proof of God"

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \cdots$$

Axiomitization of Mathematics

Why?

Guido Grandi's "Proof of God"

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \cdots$$

$$(1 - 1) + (1 - 1) + (1 - 1) + \cdots = 0$$

Axiomitization of Mathematics

Why?

Guido Grandi's "Proof of God"

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \cdots$$

$$(1 - 1) + (1 - 1) + (1 - 1) + \cdots = 0$$

$$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots = 1$$

Thus, $1 = 0$ implying we get something from nothing. Thus, ...

Why is Axiomitization Necessary?

Another "Proof" that $1 = 0$

$$\alpha = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

Why is Axiomitization Necessary?

Another "Proof" that $1 = 0$

$$\alpha = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

$$\alpha \approx 0.693$$

Why is Axiomitization Necessary?

Another "Proof" that 1 = 0

$$\alpha = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

$$\alpha \approx 0.693$$

$$\frac{1}{2} \cdot \alpha = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots$$

Why is Axiomitization Necessary?

Another "Proof" that 1 = 0

$$\alpha = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

$$\alpha \approx 0.693$$

$$\frac{1}{2} \cdot \alpha = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots$$

$$\alpha + \frac{1}{2} \cdot \alpha = 1 + \frac{1}{3} - \frac{2}{4} + \frac{1}{5} - \frac{2}{8} + \frac{1}{7} + \cdots$$

Why is Axiomitization Necessary?

Another "Proof" that $1 = 0$

$$\alpha = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

$$\alpha \approx 0.693$$

$$\frac{1}{2} \cdot \alpha = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots$$

$$\alpha + \frac{1}{2} \cdot \alpha = 1 + \frac{1}{3} - \frac{2}{4} + \frac{1}{5} - \frac{2}{8} + \frac{1}{7} + \cdots = \alpha$$

Thus, $\frac{1}{2}\alpha = 0 \Leftrightarrow 1 = 0$

Why is Axiomitization Necessary? Calculus!

Calculus was a revolutionary discovery (or invention?) by Leibniz and Newton, but at the time there were significant “leaps” of logic in their work.

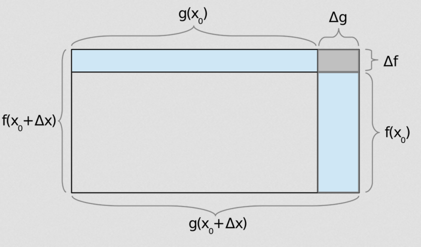
Leibiz used the notion of an *infinitesimal*, dx, which is a “number” smaller than any positive real real number to build his calculus. Mathematicians and philosophers at the time were skeptical that this new algebra of infinitesimals was logically sound.

Gottfried Leibniz

Leibniz' "Proof" of the Product Rule

Leibniz said let Δx be a real number smaller than any other real number (a "infinitesimal"). Note, in the picture below:

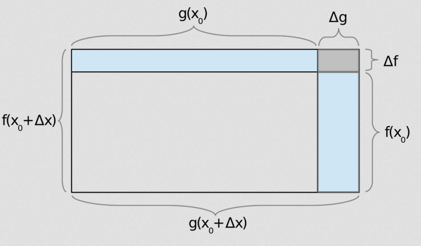
$$f(x_0 + \Delta x) \cdot g(x_0 + \Delta x) = f(x_0) \cdot g(x_0) + f(x_0) \cdot \Delta g + g(x_0) \cdot \Delta f + \Delta f \cdot \Delta g$$



Leibniz' "Proof" of the Product Rule

Leibniz said let Δx be a real number smaller than any other real number (a "infinitesimal"). Note, in the picture below:

$$f(x_0+\Delta x)\cdot g(x_0+\Delta x) = f(x_0)\cdot g(x_0) + f(x_0)\cdot \Delta g + g(x_0)\cdot \Delta f + \Delta f \cdot \Delta g$$



$$\begin{aligned}\frac{d}{dx}(f\cdot g) &= f(x_0+\Delta x)\cdot g(x_0+\Delta x) - f(x_0)\cdot g(x_0)\\ &= f(x_0)\cdot \Delta g + g(x_0)\cdot \Delta f + \Delta f \cdot \Delta g\\ &= f\cdot \frac{dg}{dx} + g\cdot \frac{df}{dx}\end{aligned}$$

Why is Axiomitization Necessary?

Also,

Why is Axiomitization Necessary?

Also, computers!!

Number systems axiomitized

Kinda Sorta History of Number Systems

- The history of the “discovery” (or is it invention?) of number systems.

$$\mathbb{N} \Rightarrow \mathbb{Z} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{R} \Rightarrow \mathbb{C}$$

Number systems axiomitized

Kinda Sorta History of Number Systems

- The history of the "discovery" (or is it invention?) of number systems.

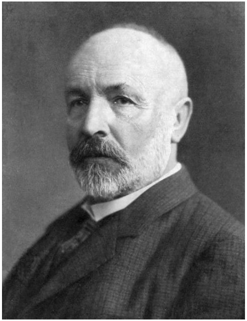
$$\mathbb{N} \Rightarrow \mathbb{Z} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{R} \Rightarrow \mathbb{C}$$

- History of the axiomitization of number systems.

$$\mathbb{C} \Rightarrow \mathbb{R} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Z} \Rightarrow \underbrace{\mathbb{N}}_{?}$$

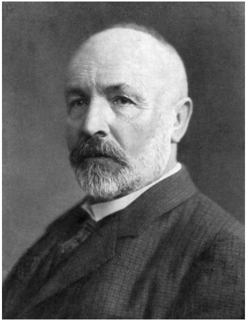
Axiomitizing the natural numbers

Georg Cantor introduced the notion of sets in the late 19th century.



Georg Cantor

Georg Cantor introduced the notion of sets in the late 19th century.



Georg Cantor

Unfortunately, some logical inconsistencies with Cantor's (Naive) Set Theory were discovered.

Issues with Self-referencing

Russell's Paradox

Let X denote the set that contains all sets, except the those sets that do not contain themselves.

$$X = \{Y : Y \notin Y\}$$

Does X contain itself?

Issues with Self-referencing

Russell's Paradox

Let X denote the set that contains all sets, except the those sets that do not contain themselves.

$$X = \{Y : Y \notin Y\}$$

Does X contain itself?

Barber of Seville

The town of Seville has just one barber. This barber is a man who shaves all those, and only those, men in town who do not shave themselves. Does the barber shave himself?

Another attempt to axiomitize the natural numbers

Principia Mathematica

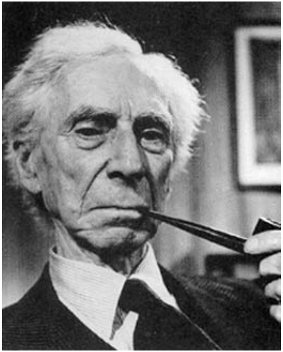
Alfred North Whitehead and Bertrand Russell published the *Principia Mathematica* or *PM* in the early 20th century that gives a set of axioms, symbols, and rules of inference from which all arithmetical truths about whole numbers could be proven......

Another attempt to axiomitize the natural numbers

Principia Mathematica

Alfred North Whitehead and Bertrand Russell published the *Principia Mathematica* or *PM* in the early 20th century that gives a set of axioms, symbols, and rules of inference from which all arithmetical truths about whole numbers could be proven......was their intent.

Bertrand Russell



Alfred North Whitehead

Snippets of *PM*

Symbols in *PM*

- $\exists$ is the symbol for the existential quantifier (there exists).
- $\forall$ is the symbol universal quantifier (for all).
- $\wedge$ is the symbol for "and", so $P \wedge Q$ represents "P and Q".
- $\sim$ is the symbol for "not" or negation.

Snippets of *PM*

Symbols in *PM*

- $\exists$ is the symbol for the existential quantifier (there exists).
- $\forall$ is the symbol universal quantifier (for all).
- $\wedge$ is the symbol for "and", so $P \wedge Q$ represents "P and Q".
- $\sim$ is the symbol for "not" or negation.
- $P \to Q$ is the symbol for the conditional statement "if P then Q."
- There are variables for numbers w, x, y, z and variables for sentences P, Q, R, S.
- Punctuation is defined such as parenthesis, commas, periods, etc.

Snippets of *PM*

Symbols in *PM*

- $\exists$ is the symbol for the existential quantifier (there exists).
- $\forall$ is the symbol universal quantifier (for all).
- $\wedge$ is the symbol for "and", so $P \wedge Q$ represents "P and Q".
- $\sim$ is the symbol for "not" or negation.
- $P \to Q$ is the symbol for the conditional statement "if P then Q."
- There are variables for numbers w, x, y, z and variables for sentences P, Q, R, S.
- Punctuation is defined such as parenthesis, commas, periods, etc.

Axioms

- There is a set that contains nothing, denoted $\emptyset$.
- For all natural numbers x, if $x = y$ then $y = x$.

Snippets of *PM*

Rules of Inference

- Modus Ponens: If "$P \to Q$" is a true statement in PM and P is true in PM, then Q is a true statement in PM.
- Modus Tollens: If "$P \to Q$" is a true statement in PM and Q is false in PM, then P is false in PM.

Snippets of *PM*

Rules of Inference

- Modus Ponens: If "$P \to Q$" is a true statement in PM and P is true in PM, then Q is a true statement in PM.
- Modus Tollens: If "$P \to Q$" is a true statement in PM and Q is false in PM, then P is false in PM.

Theorems and Proofs

A theorem is a sentence in PM that has been proven using the axioms and rules of inferences. A proof is a finite string of sentences using the rules of inferences, axioms, and theorems.

Examples

Consider the following two sentences in *PM*

1. $$S_1 : (\forall y)(\exists x)(y < x)$$

2. $$S_2 : (\forall y)(\exists x)\left(y = x^2\right)$$

Examples

Consider the following two sentences in *PM*

1. $$S_1 : (\forall y)(\exists x)(y < x)$$

2. $$S_2 : (\forall y)(\exists x)\left(y = x^2\right)$$

Translation of these statements:

1. S_1 says for every whole number there is a whole number bigger than that number. (True)
2. S_2 says every real number is a perfect square. (False, $y = 3$)

Is *PM* Complete and Consistent?

Is *PM* consistent?

Suppose there was some sentence T in PM such that T and $\sim T$ (the negation of T) can be proven within PM?
If there exists is such a sentence, we would say that PM is inconsistent.

Is *PM* Complete and Consistent?

Is *PM* consistent?

Suppose there was some sentence T in PM such that T and $\sim T$ (the negation of T) can be proven within PM?
If there exists is such a sentence, we would say that PM is inconsistent.

Is *PM* complete?

We say that PM is complete if either S or $\sim S$ can be proven **within** PM for any any sentence S that asserts some property about whole numbers.

Proving statements within the system

Jumping out of the System

Let T be the sentence $[(\forall n \geq 3)\,(\exists x)(\exists y)(\exists z)(x^n + y^n = z^n)]$.

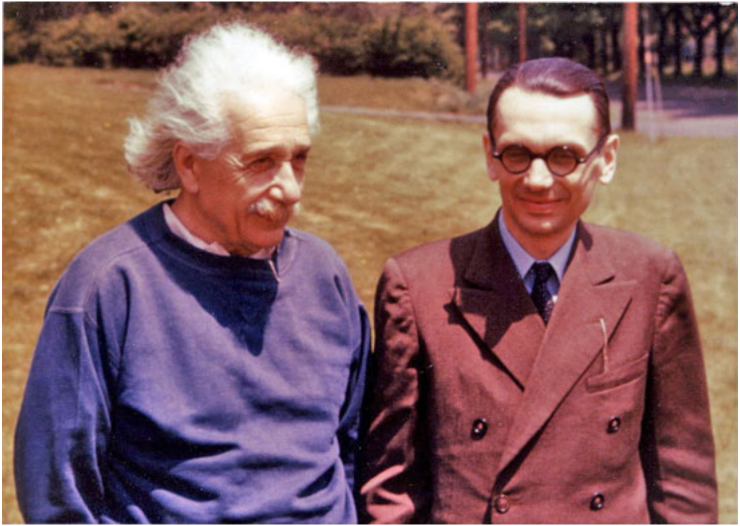
Proving statements within the system

Jumping out of the System

Let T be the sentence $[(\forall n \geq 3)\,(\exists x)(\exists y)(\exists z)(x^n + y^n = z^n)]$.

- $\sim T$ is Fermat's Last Theorem. It was first posed by Fermat in 1637.
- It was finally proven by Andrew Wiles in 1994.
- Wiles' proof of Fermat's Last Theorem does not stay within *PM*.

Kurt Gödel



Richard's Paradox by Jules Richard

- Given a language (English) that can express purely arithmetical properties of whole numbers such as "an integer is divisible by 10" and "an integer is the product of two integers", etc.

- These properties can be placed in serial order: property *a* precedes property *b* if it has fewer letters than *b* or if they have the same number of letters and *a* precedes *b* alphabetically.

- List these properties in order, so there is a unique integer the corresponds to each property.

Richard's Paradox

- We say a whole number n is said to be *Richardian* if n does **not** have the property designated by its corresponding arithmetical property.
- Example: Suppose the property that corresponds to 17 is "not divisible by any other integer other than 1 or itself". Then 17 is not Richardian.

Richard's Paradox

- We say a whole number n is said to be *Richardian* if n does **not** have the property designated by its corresponding arithmetical property.
- Example: Suppose the property that corresponds to 17 is "not divisible by any other integer other than 1 or itself". Then 17 is not Richardian.
- The property of whether a number is Richardian is a property of whole numbers, so it is on the list somewhere and has a corresponding number r.
- Is r Richardian?

Gödel modeled his proof in his 1931 paper "On Formally Undecidable Propositions of Principia Mathematica and Related Systems" on Richard's Paradox, but he was able to circumvent the logical flaws Richard's paradox.

What exactly did he do?

He started with what is now called Gödel numbering.

Gödel's Incompleteness Theorems

- Gödel showed how to construct a formula G in PM that represents the meta-mathematical statement "The formula G is not demonstrable within PM." G is called the Gödel Formula.

Gödel's Incompleteness Theorems

- Gödel showed how to construct a formula G in PM that represents the meta-mathematical statement "The formula G is not demonstrable within PM." G is called the Gödel Formula.

- Gödel then showed that G is demonstrable in PM if and only if $\sim G$ is demonstrable in PM. Thus, he showed that if PM is consistent, then it is incomplete.

- Gödel used a meta-mathematical argument to show that G is a true mathematical formula. Thus, PM is incomplete.

Gödel's Incompleteness Theorems

- Gödel also showed that PM is essentially incomplete. Suppose we take G to be a new axiom in PM. In Gödel's proof, he showed how to construct a new Gödel formula G' in this "new" system.

- Moreover, his construction is recursive, so for any finitely many axioms are added to the list, a new Gödel formula can always be constructed in the new system.

Gödel's Incompleteness Theorems

- Gödel also showed that *PM* is essentially incomplete. Suppose we take G to be a new axiom in *PM*. In Gödel's proof, he showed how to construct a new Gödel formula G' in this "new" system.
- Moreover, his construction is recursive, so for any finitely many axioms are added to the list, a new Gödel formula can always be constructed in the new system.
- Finally, he showed that his construction can be done in **any** axiomatic system that is strong enough to talk about the arithmetic of whole numbers.

Implications for Physics

nature International weekly journal of science

Home | News & Comment | Research | Careers & Jobs | Current Issue | Archive | Audio & Vid

News & Comment > News > 2015 > December > Article

NATURE | NEWS

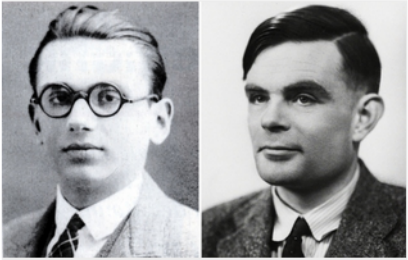
Paradox at the heart of mathematics makes physics problem unanswerable

Gödel's incompleteness theorems are connected to unsolvable calculations in quantum physics.

Davide Castelvecchi

09 December 2015

Rights & Permissions



Kurt Gödel (left) demonstrated that some mathematical statements are undecidable; Alan Turing (right) connected that proof to unsolvable algorithms in computer science.

Lucas-Penrose Argument Against Artificial Intelligence Using Gödel's Incompleteness Theorems

Any computer or Turing machine is built upon a finite set of instructions. If such a machine were programmable to perform operations in PM, then Gödel showed that the machine will have certain limitations that the human mind will not, such as the Gödel formula G. Such a machine can only "know" what it can prove using its processors, rules of inference, etc, so it will have a Gödel formula, a statement that is true, but it cannot see to be true.

Implications for Artificial Intelligence

What follows is an argument against Lucas-Penrose's argument against artificial intelligence using Gödel's Incompleteness Theorems by using Gödel's Incompleteness Theorems.

- A single person (let's say Leonard Euler) will have a finite number of symbolic thoughts in their lifetime.
- Thus, the set of all symbolic mathematical truth that Euler will ever know is finite, and thus it is axiomitizable.

Limits to Humanly Knowable Mathematics

- Surely, Euler could do arithmetic and even prove basic statements about whole numbers. In other words the "system" that is Euler is strong enough to talk about arithmetic.

Limits to Humanly Knowable Mathematics

- Surely, Euler could do arithmetic and even prove basic statements about whole numbers. In other words the "system" that is Euler is strong enough to talk about arithmetic.

- Thus, by Gödel's Incompleteness Theorems, there is a statement about whole numbers G_E that is true, but is not demonstrable by Euler. Euler has a Gödel formula.

- In essence, the brain or consciousness of Euler has limitations just as a Turing machine does.

Limits to Humanly Knowable Mathematics

- The number of symbolic thoughts of any one person is finite. Thus, at any moment in history, the total number of symbolic thoughts by all humans is still finite and hence axiomitizable.

- Thus, humanity as a whole (at any moment in time) has a Gödel formula, G_H, a statement about whole numbers that while true, we can never reach.

- The number of symbolic thoughts of any one person is finite. Thus, at any moment in history, the total number of symbolic thoughts by all humans is still finite and hence axiomitizable.

- Thus, humanity as a whole (at any moment in time) has a Gödel formula, G_H, a statement about whole numbers that while true, we can never reach.

- Then there is at least one true statement about whole numbers that humanity will never be able to prove.

Thank you!

Questions?

Comments?

Discussion?

Bibliography

By request. I do have one.