Randomness

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In Theory and Practice

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CMC3 Conference
Monterey, Dec 2014

Randomness

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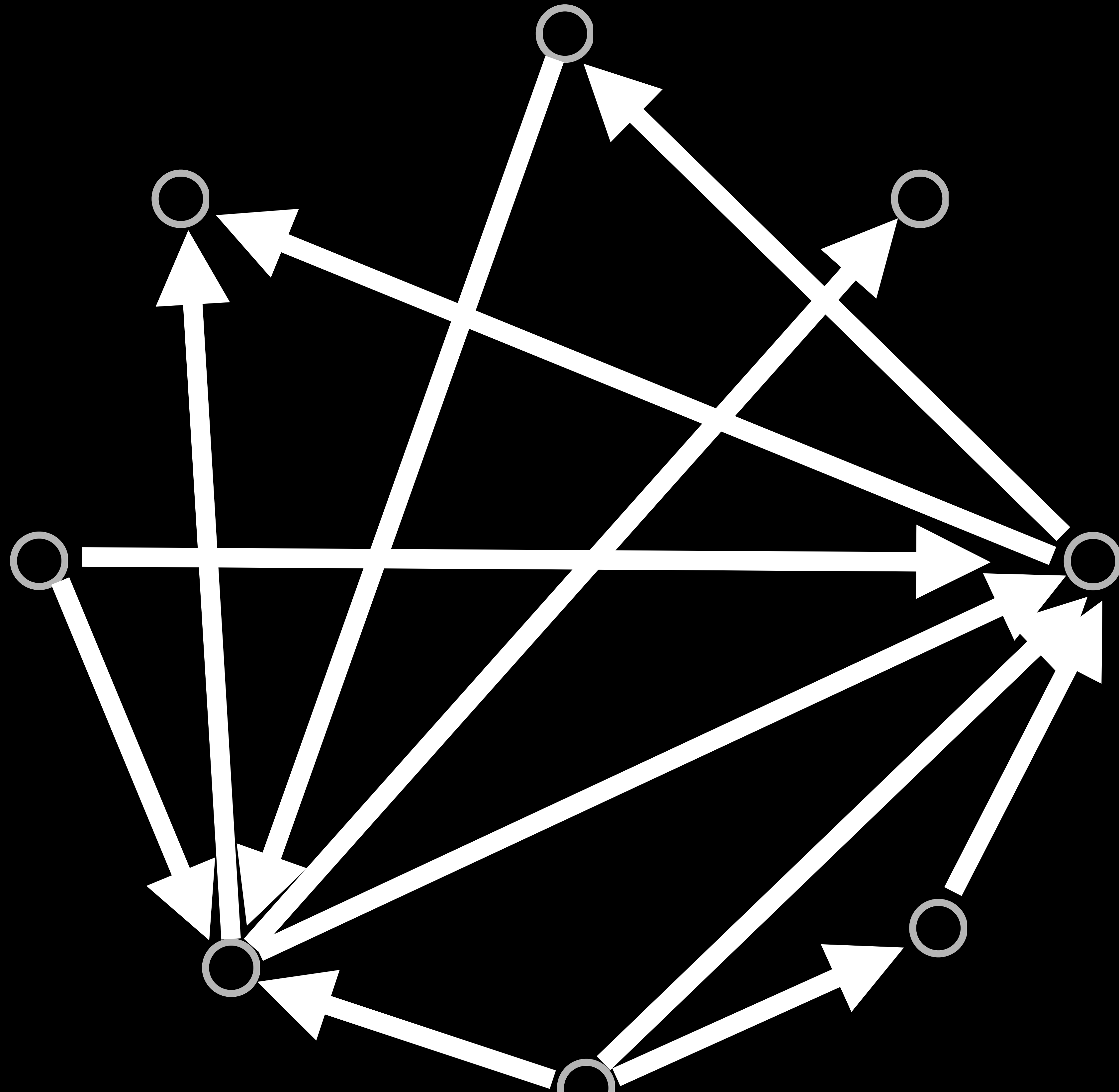
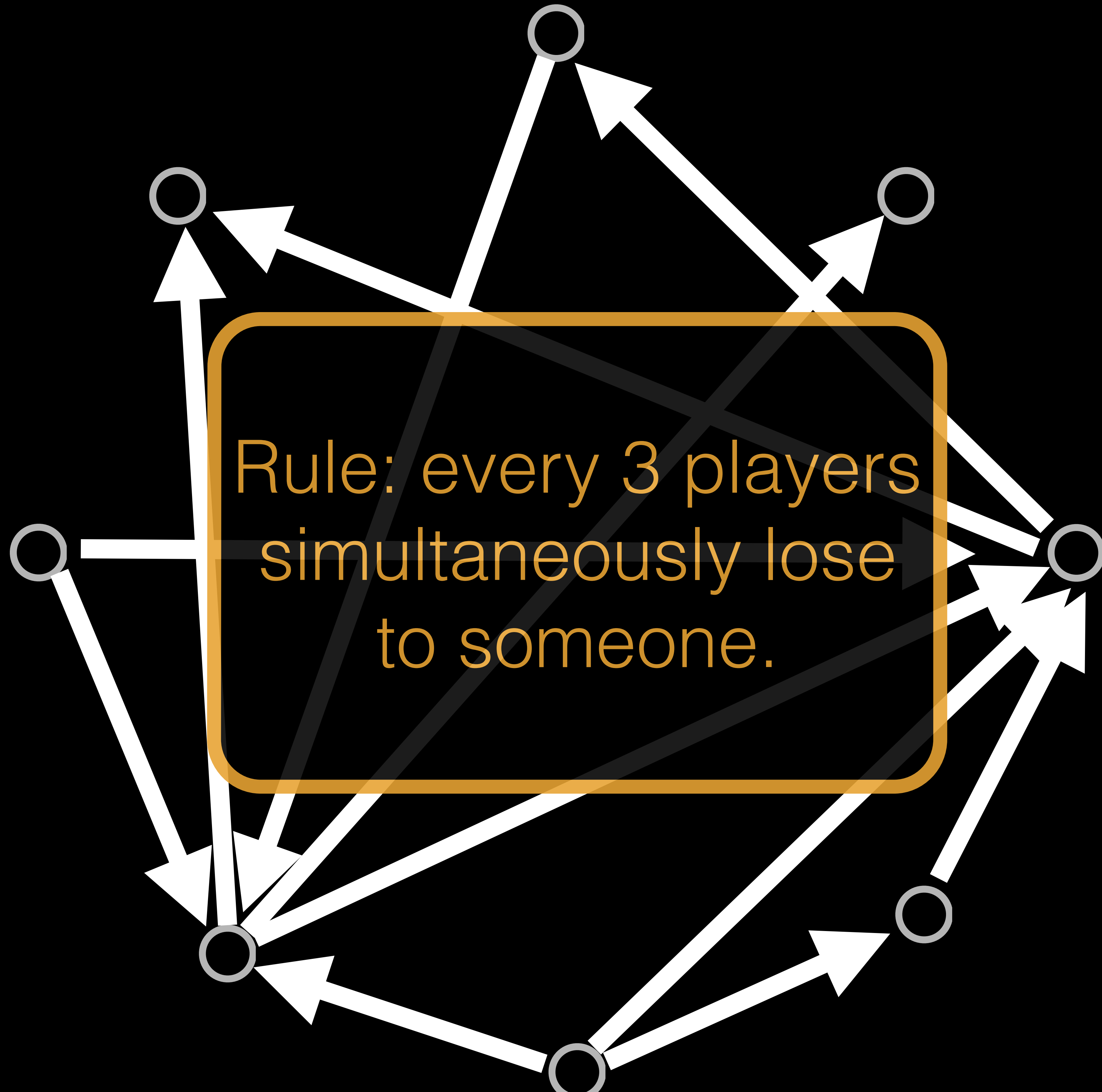
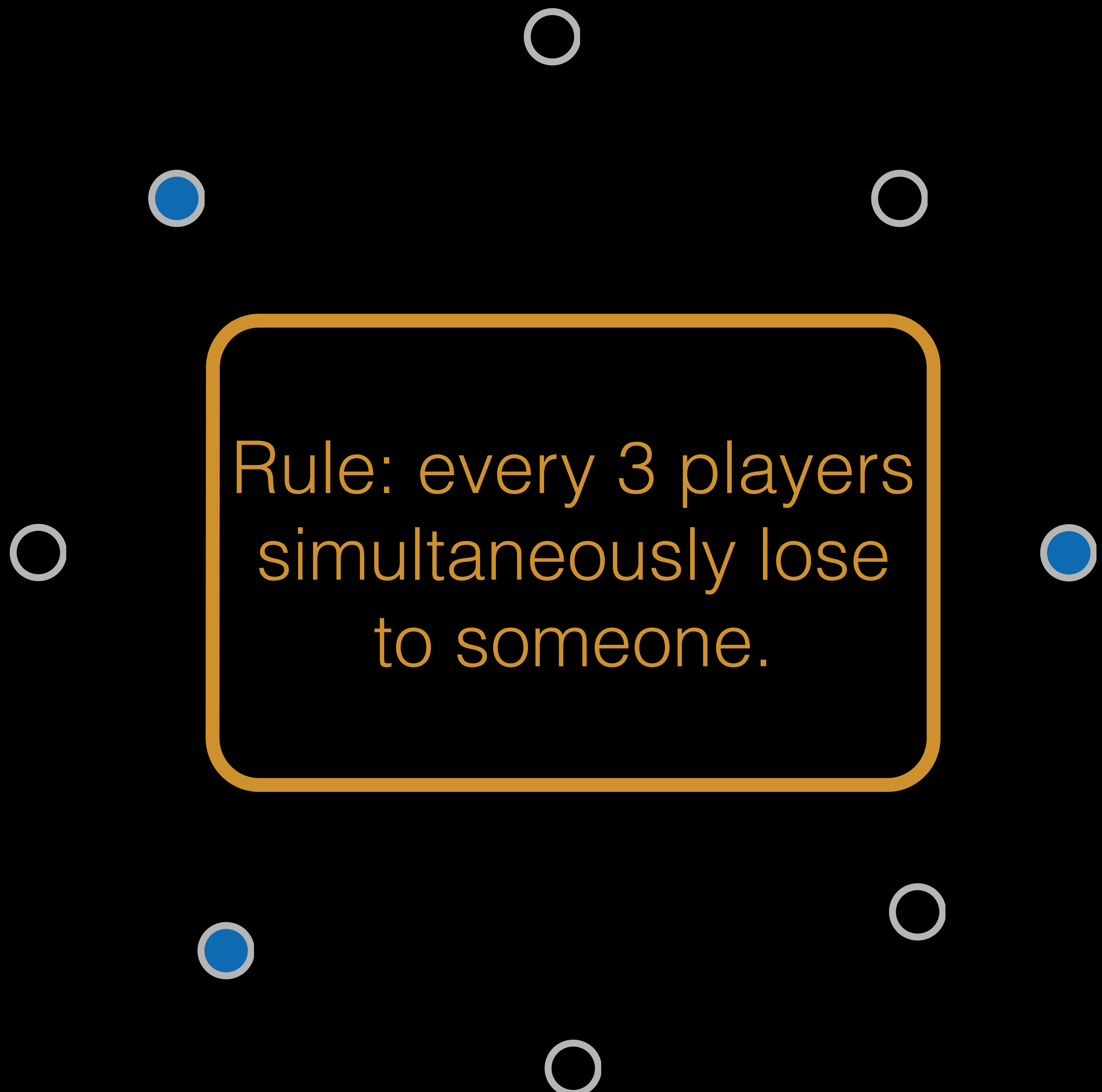
♥

Mathematics

1. Tournaments




Rule: every 3 players simultaneously lose to someone.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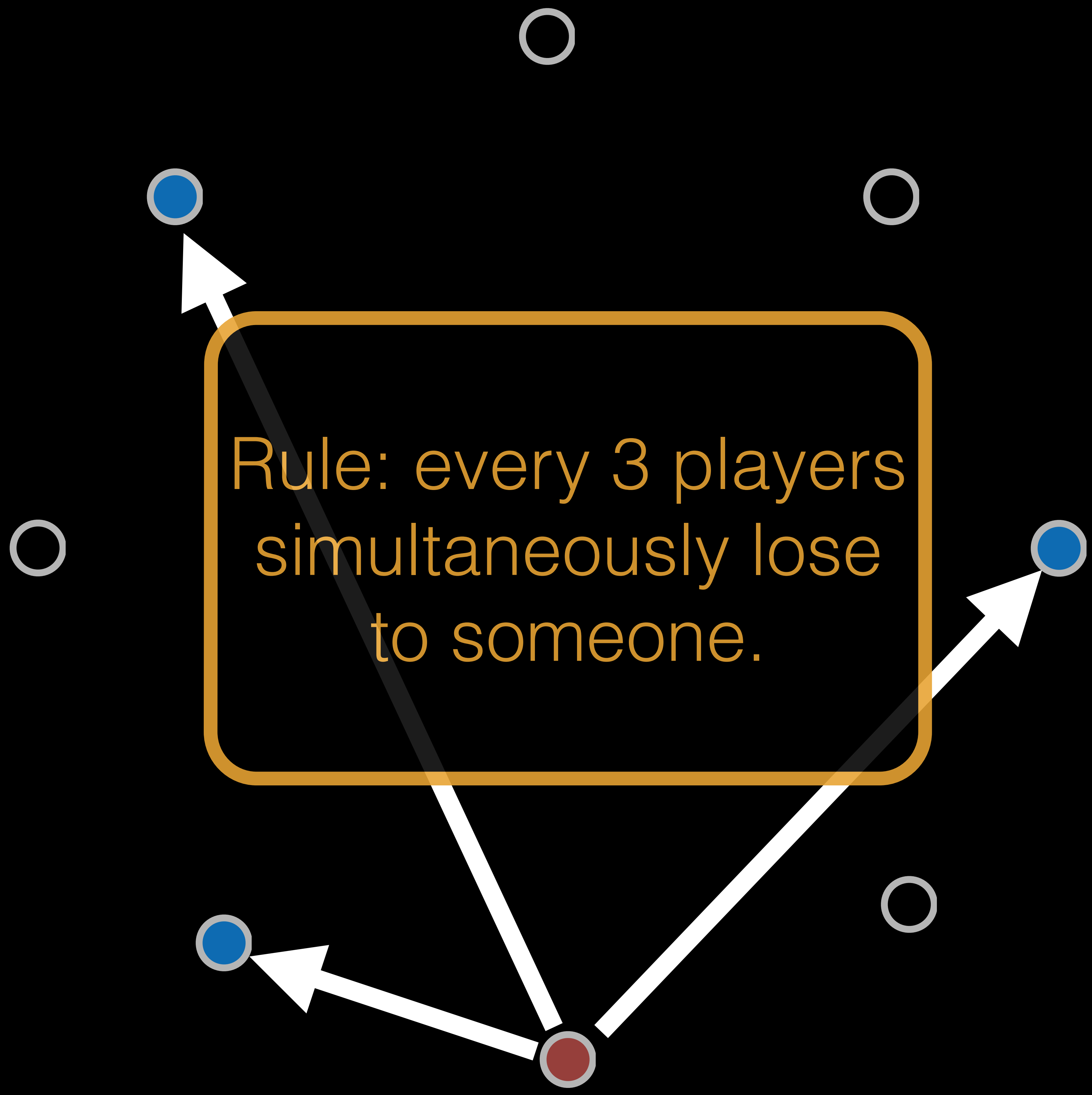
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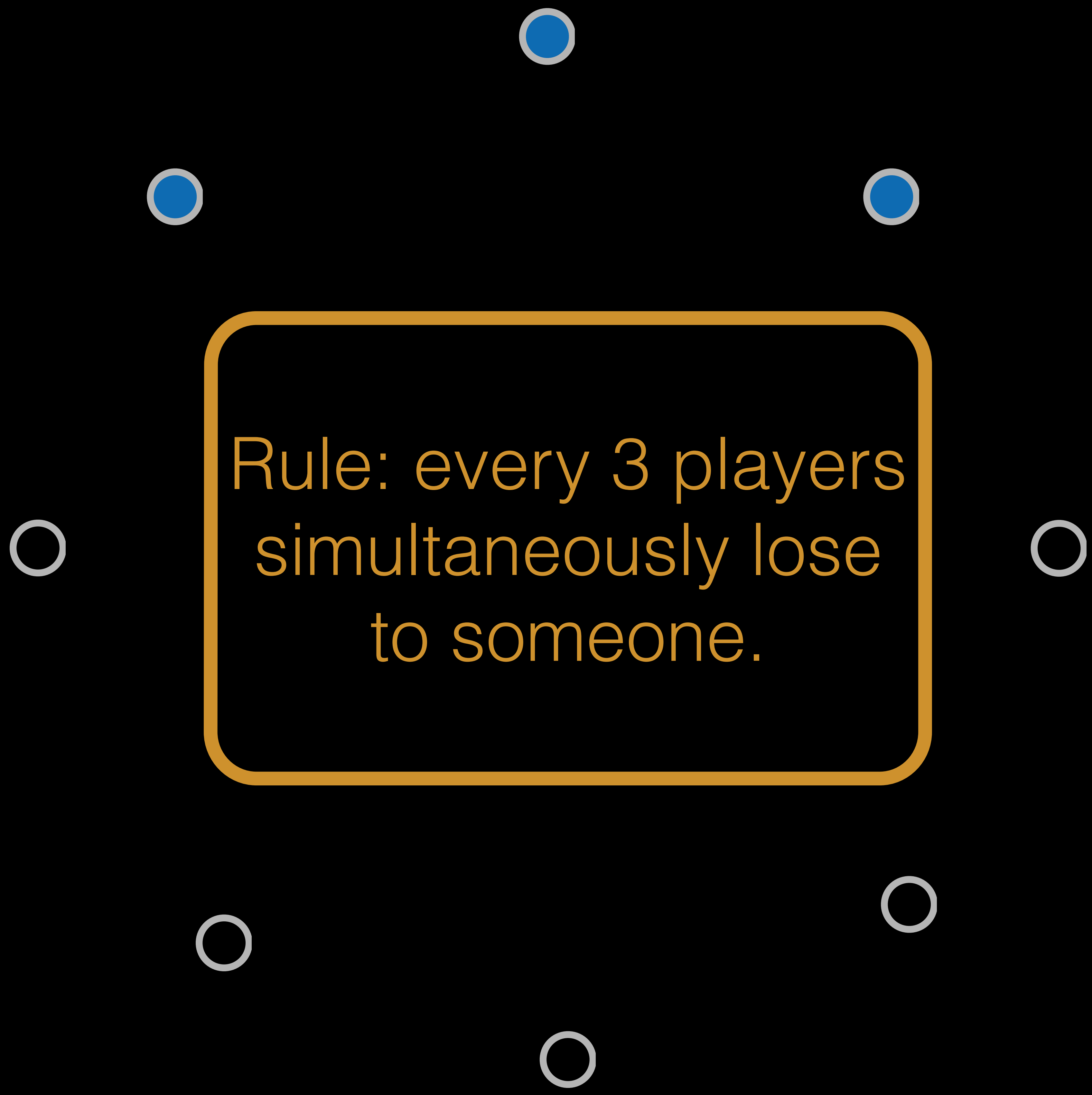
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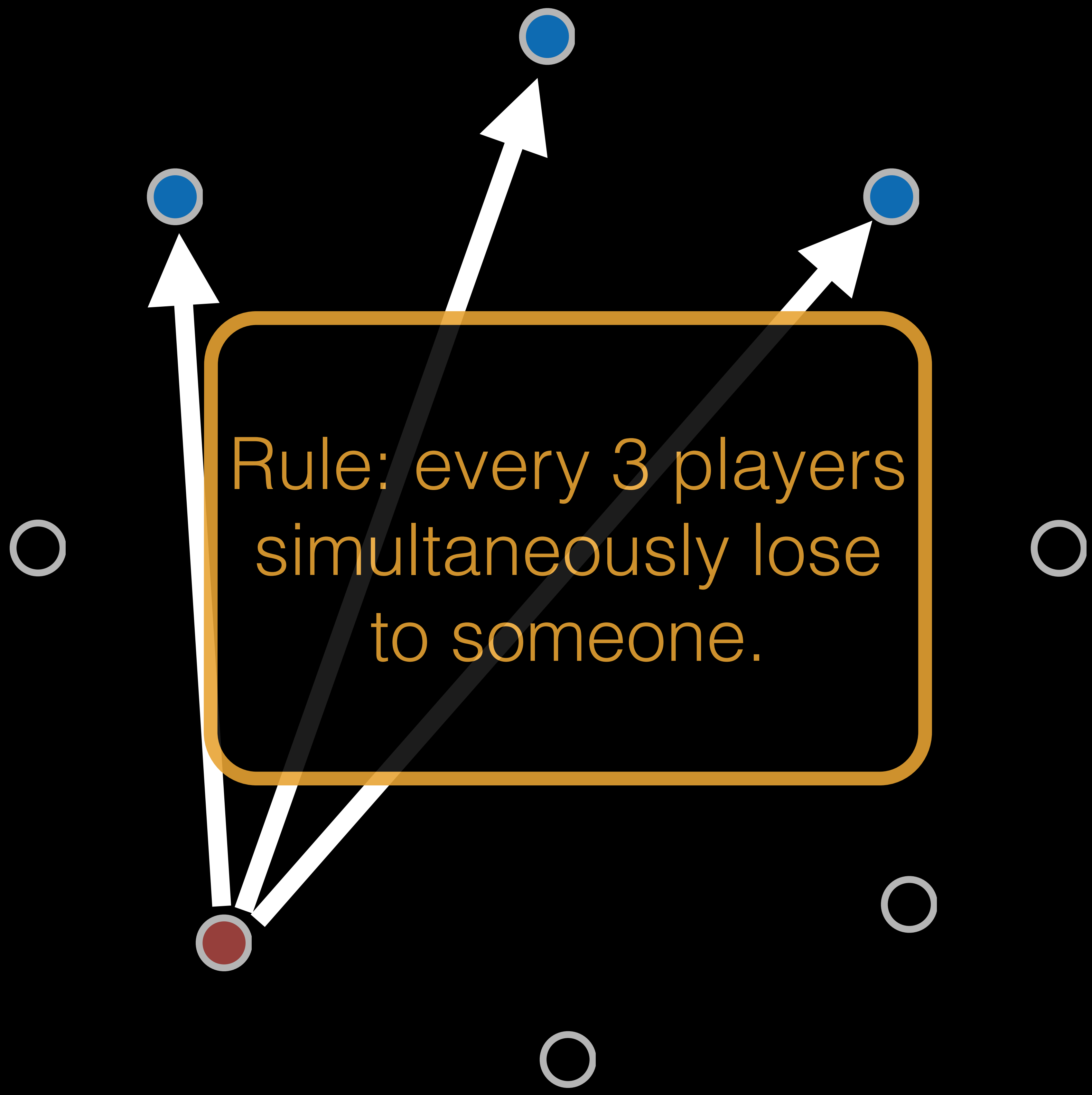
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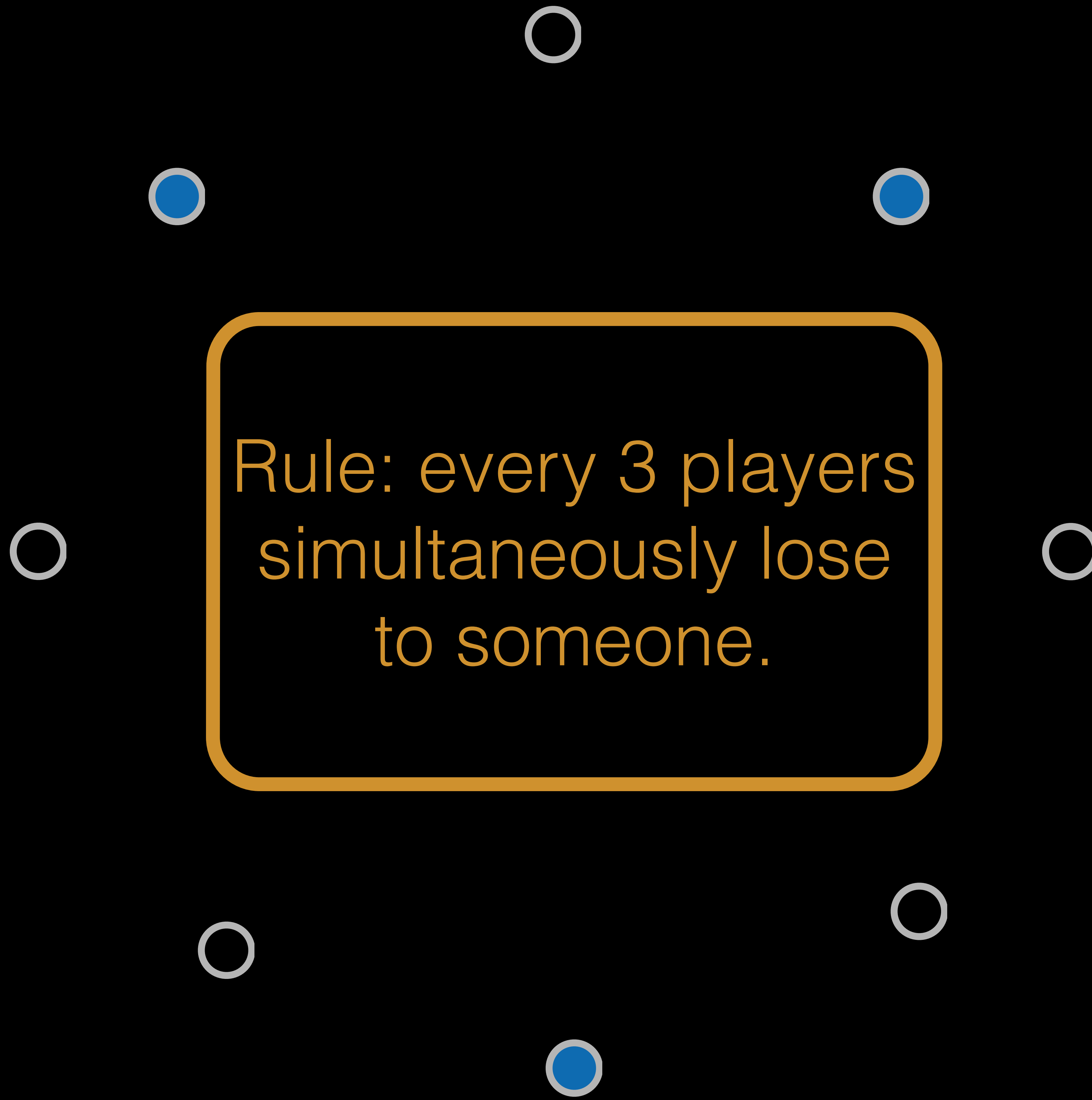
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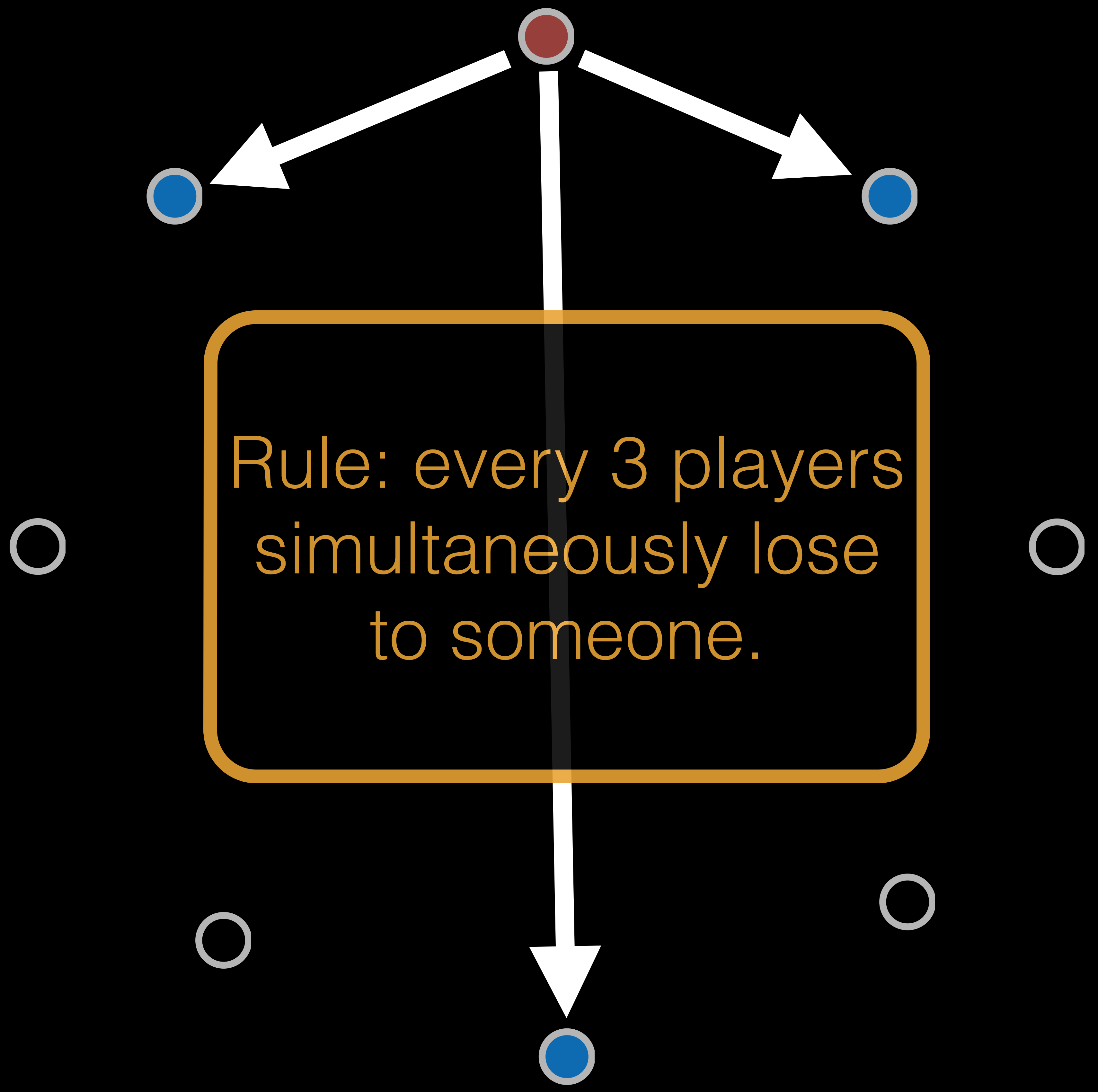
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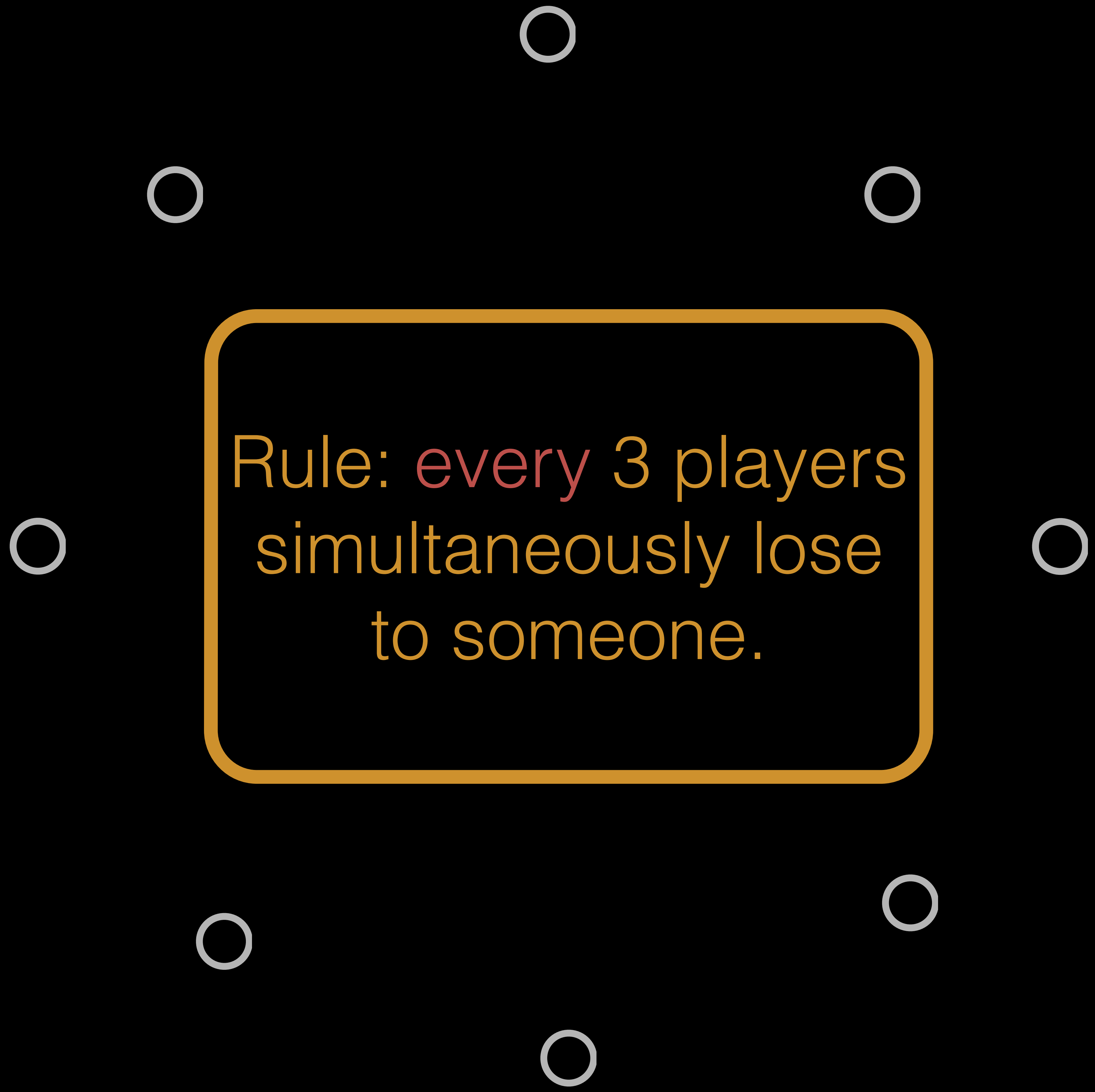
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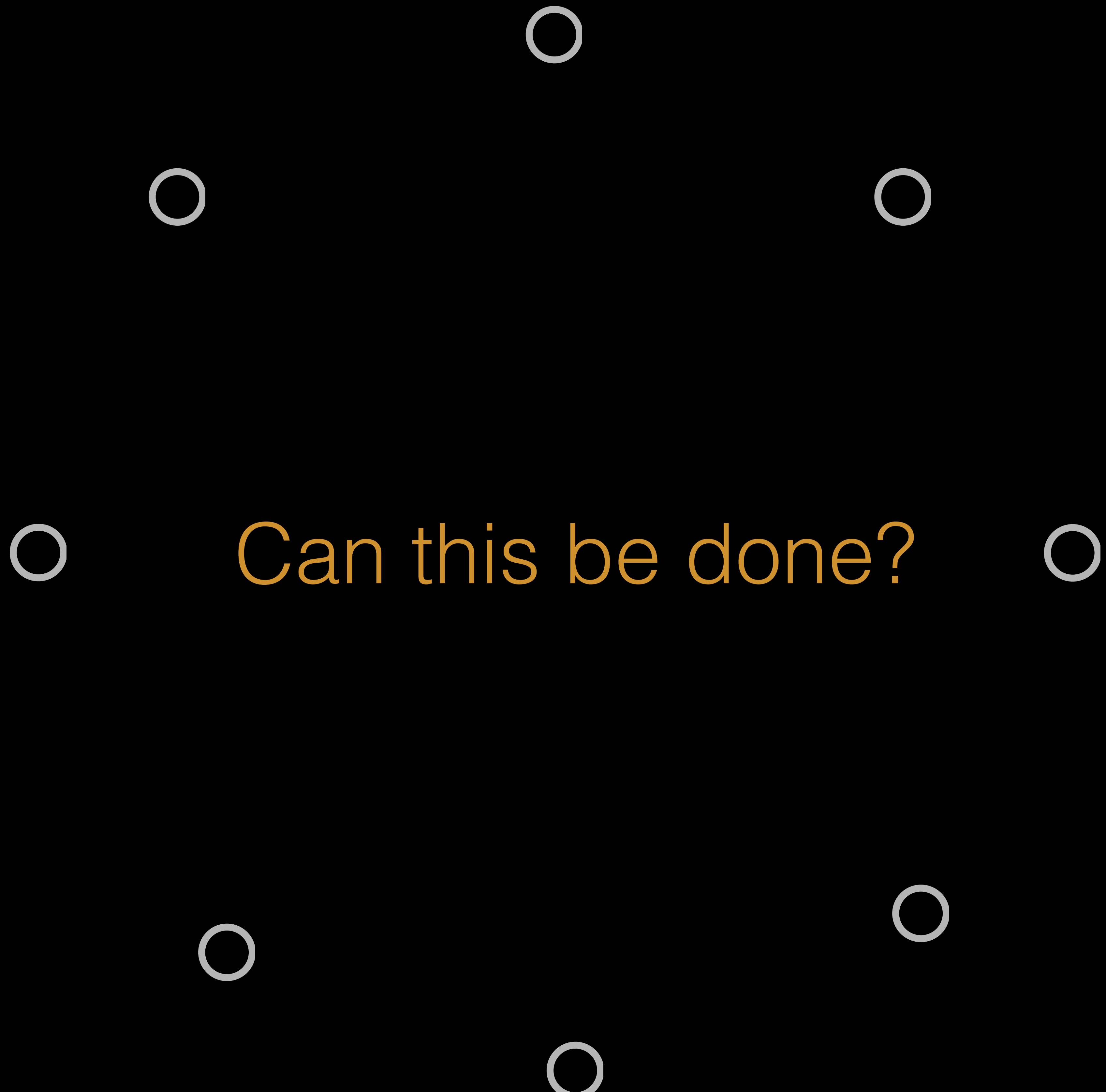

Can this be done?

YES.

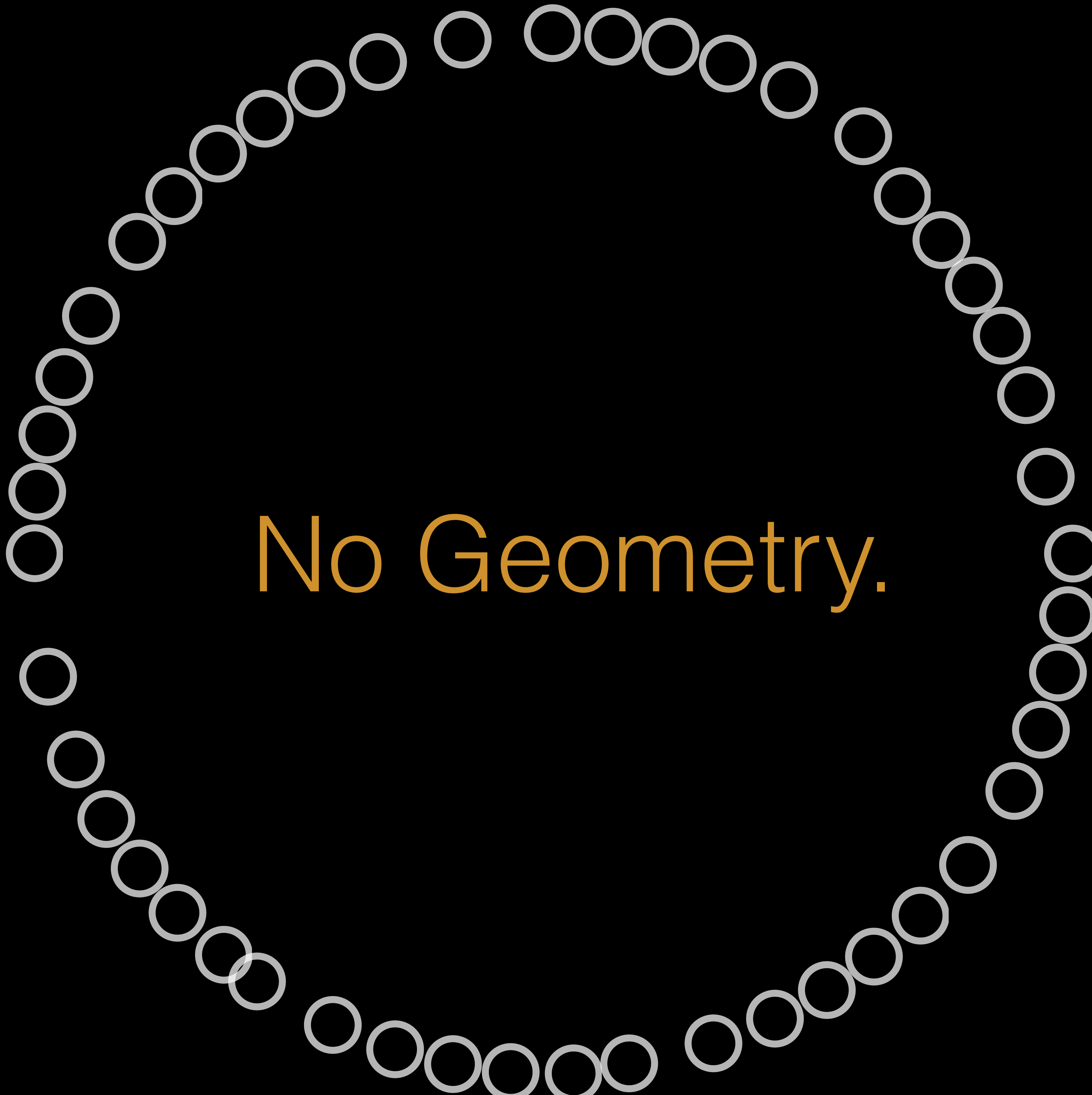
Actually…

No.

We need 100 Players.

But How?

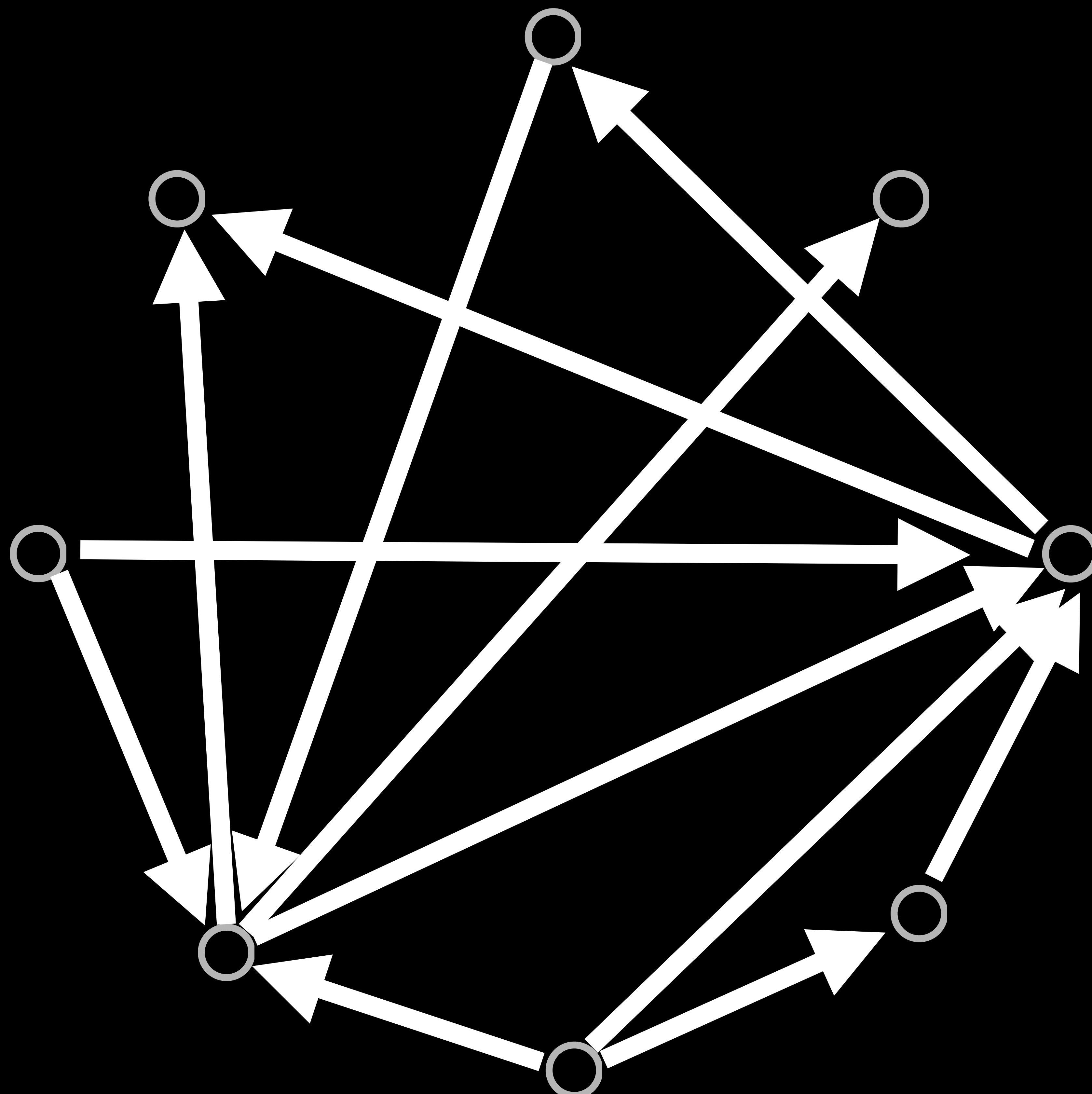
No Algebra.

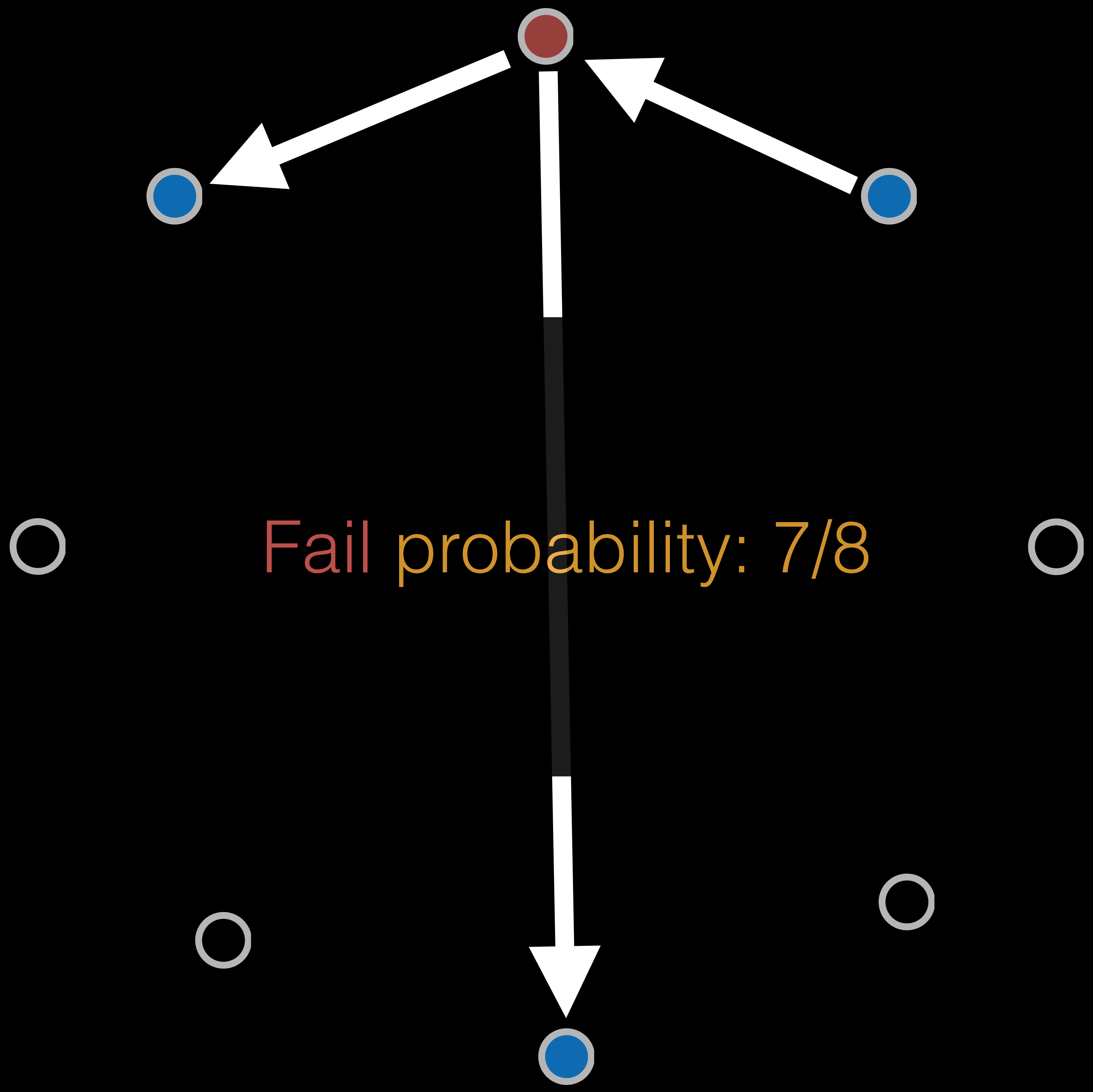

No Geometry.

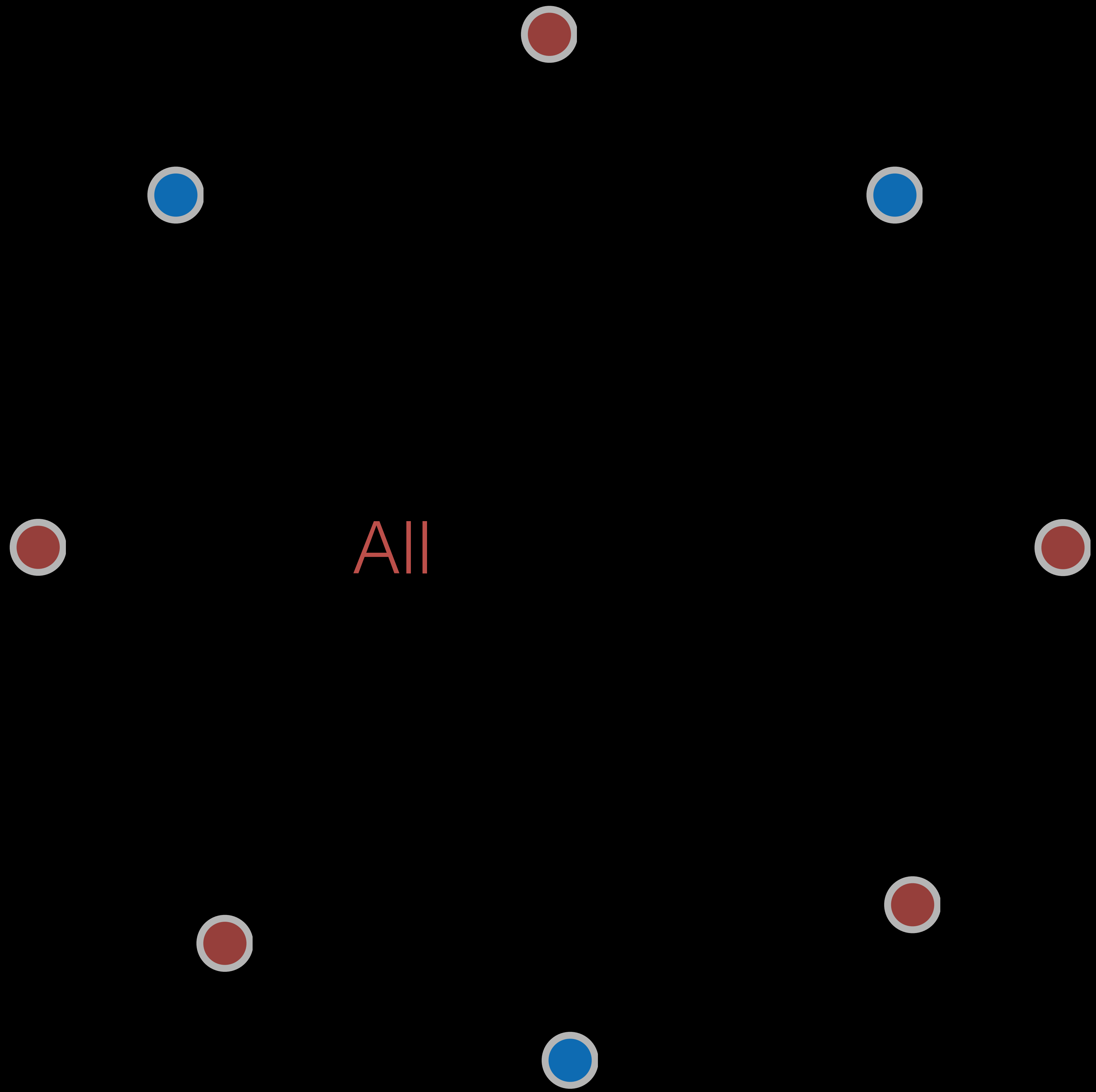
No Rules.

No Thought.

Just choose each arrow at random.



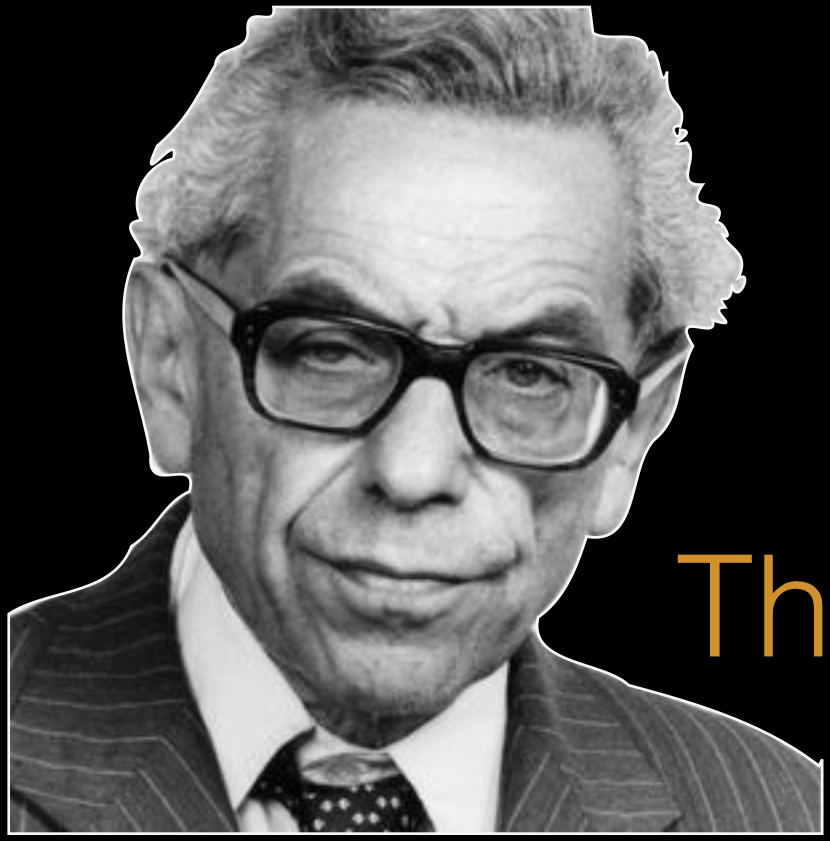

Fail probability: 7/8


All

Total Failed Tournaments:

$$\binom{100}{3}\left(\frac{7}{8}\right)^{97}$$

At least 2/3 of those random assignments succeed.



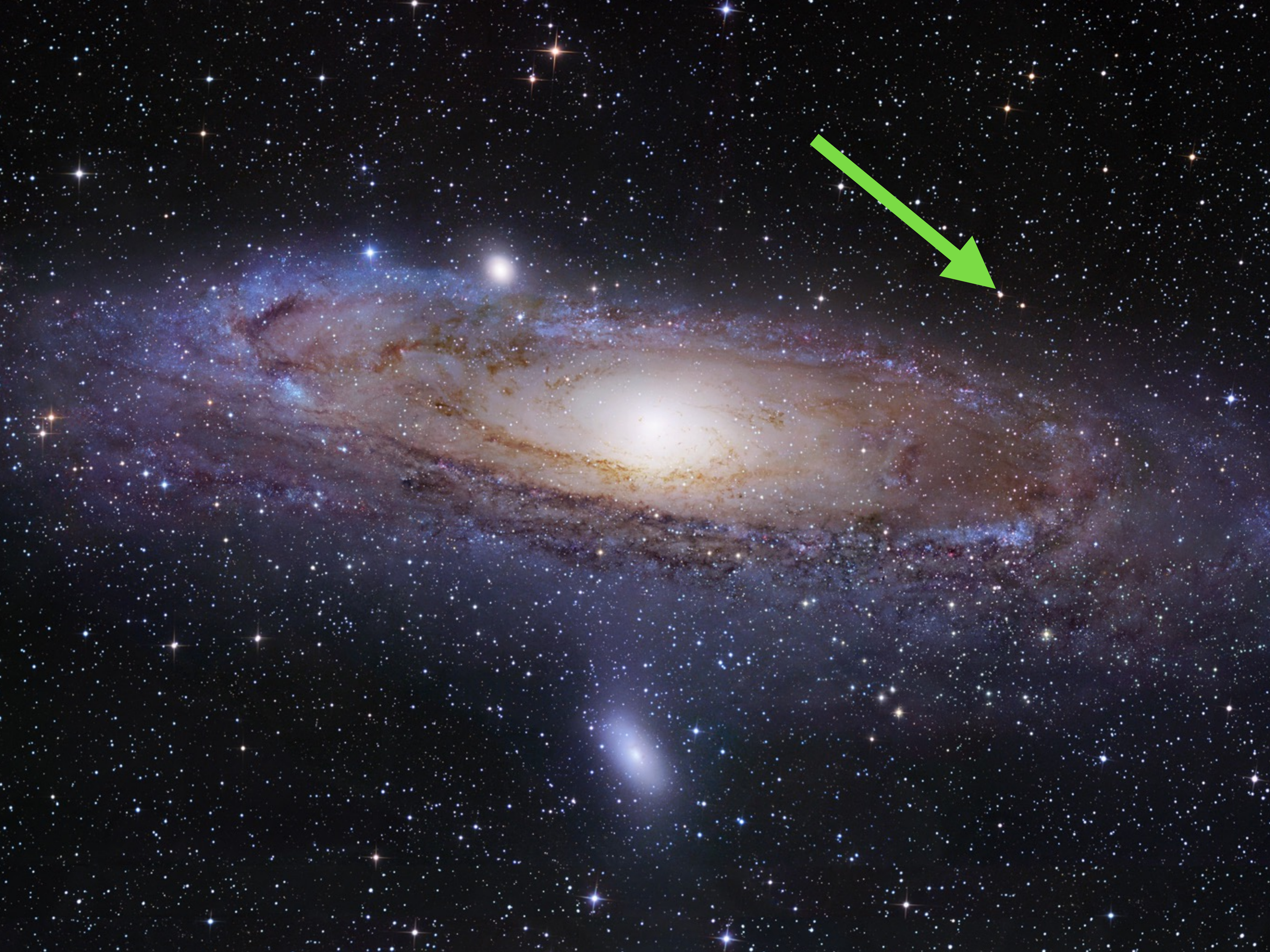
The

Method

Sometimes it's easier to randomly choose than to explicitly construct.

Often, 99.9% of the objects have what you need…

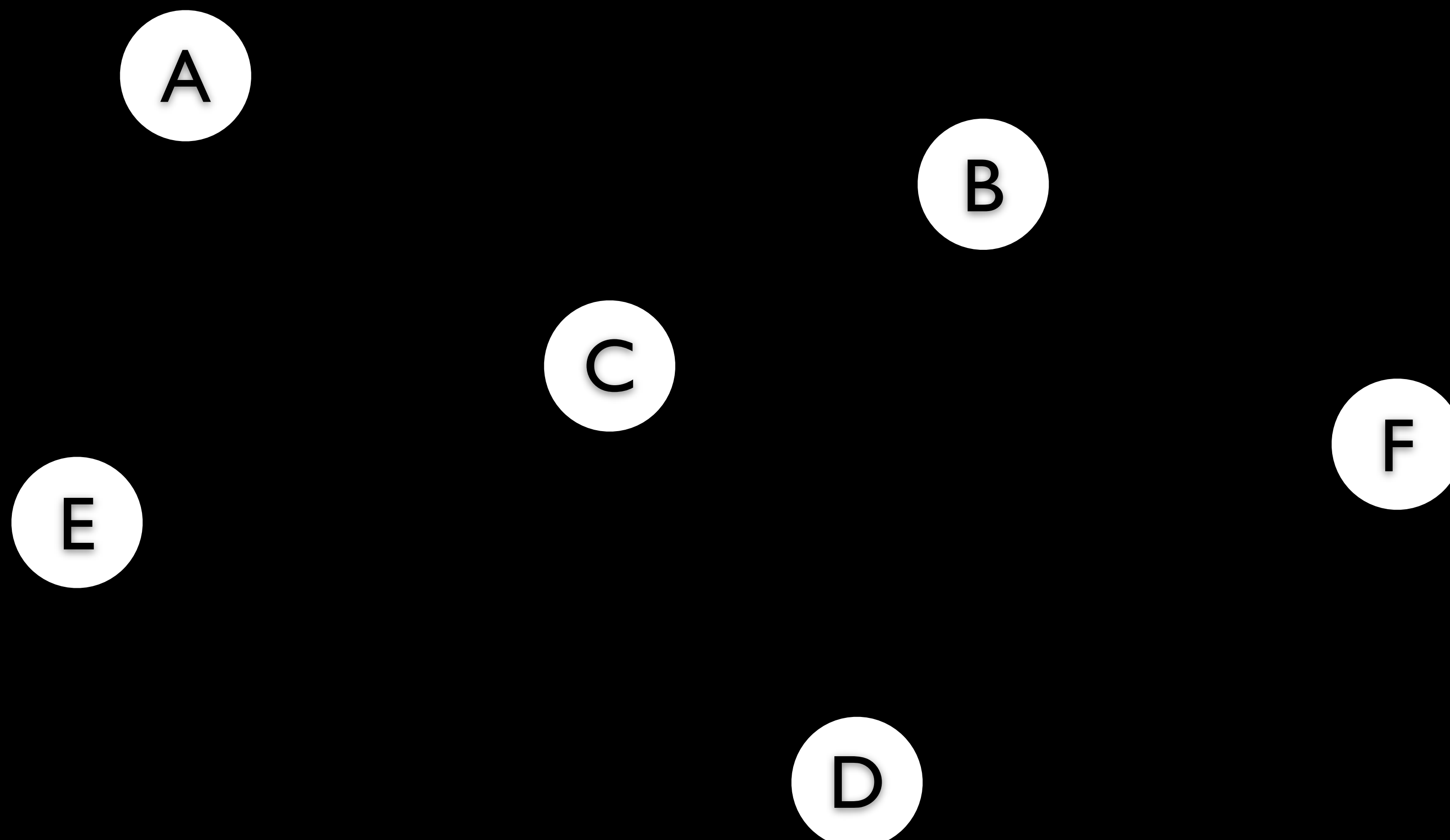
…but you can’t find them!



2. High Girth and Chromatic Number

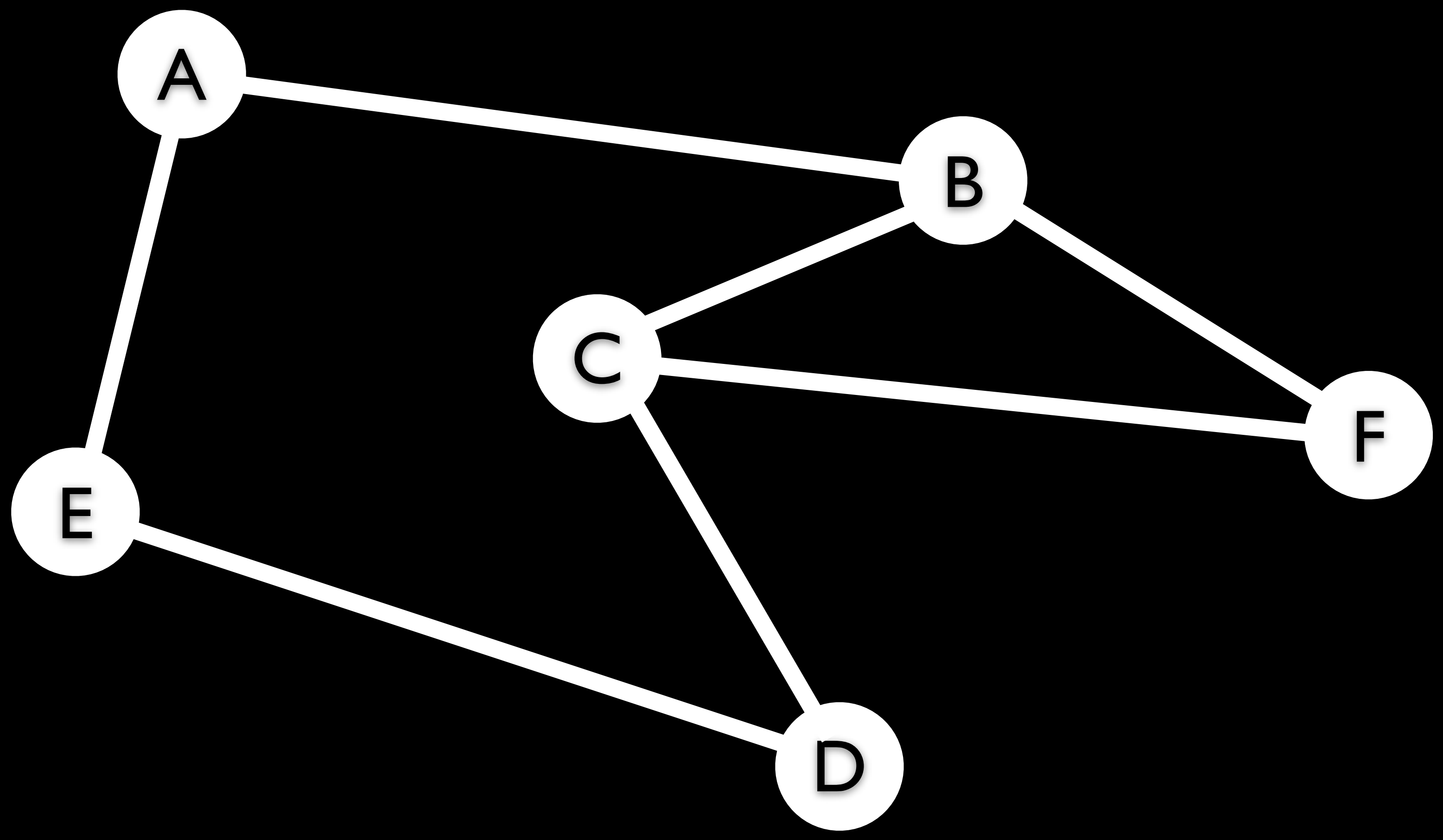
Graphs

Graphs



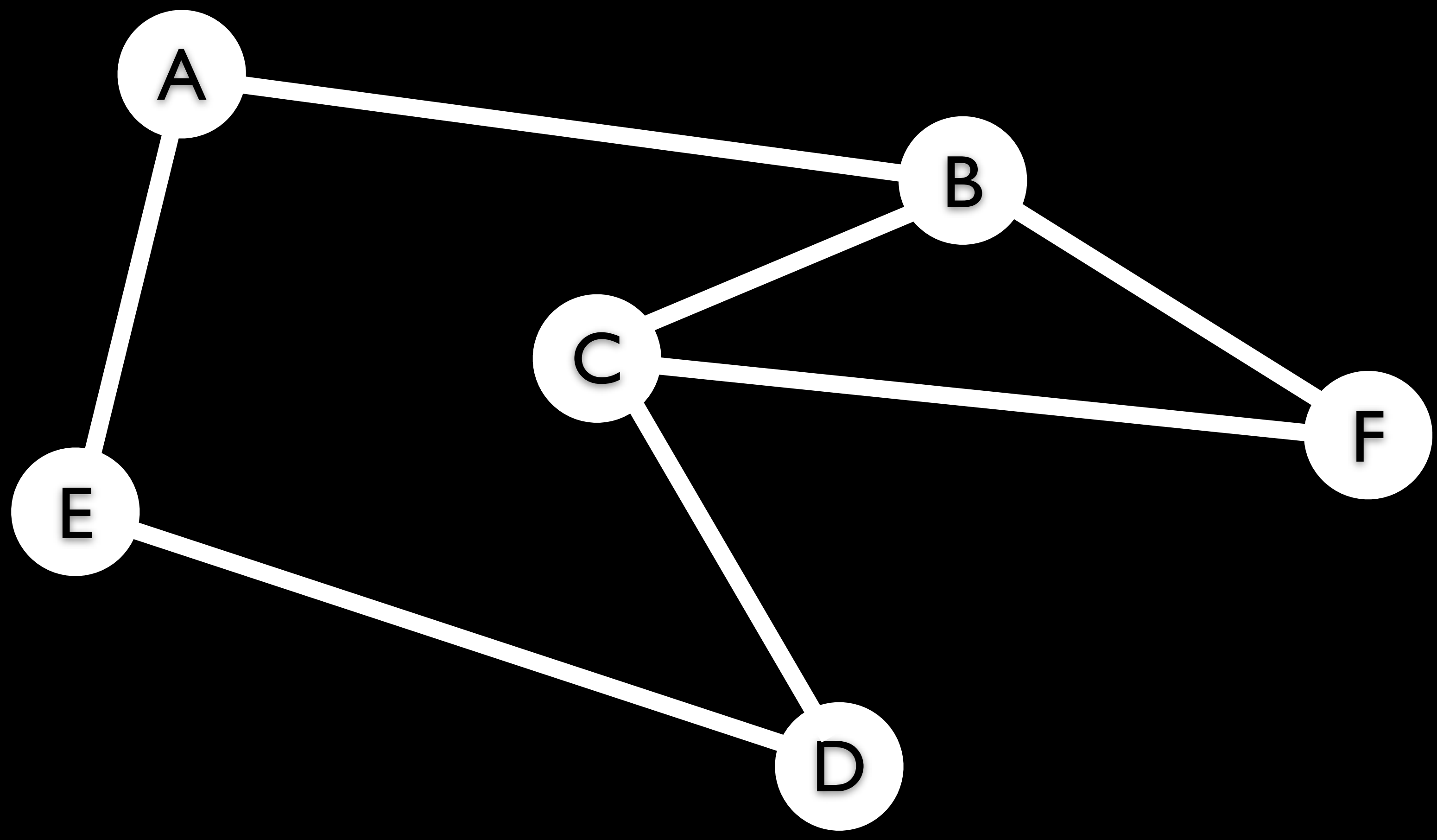
Vertices

Graphs



Edges

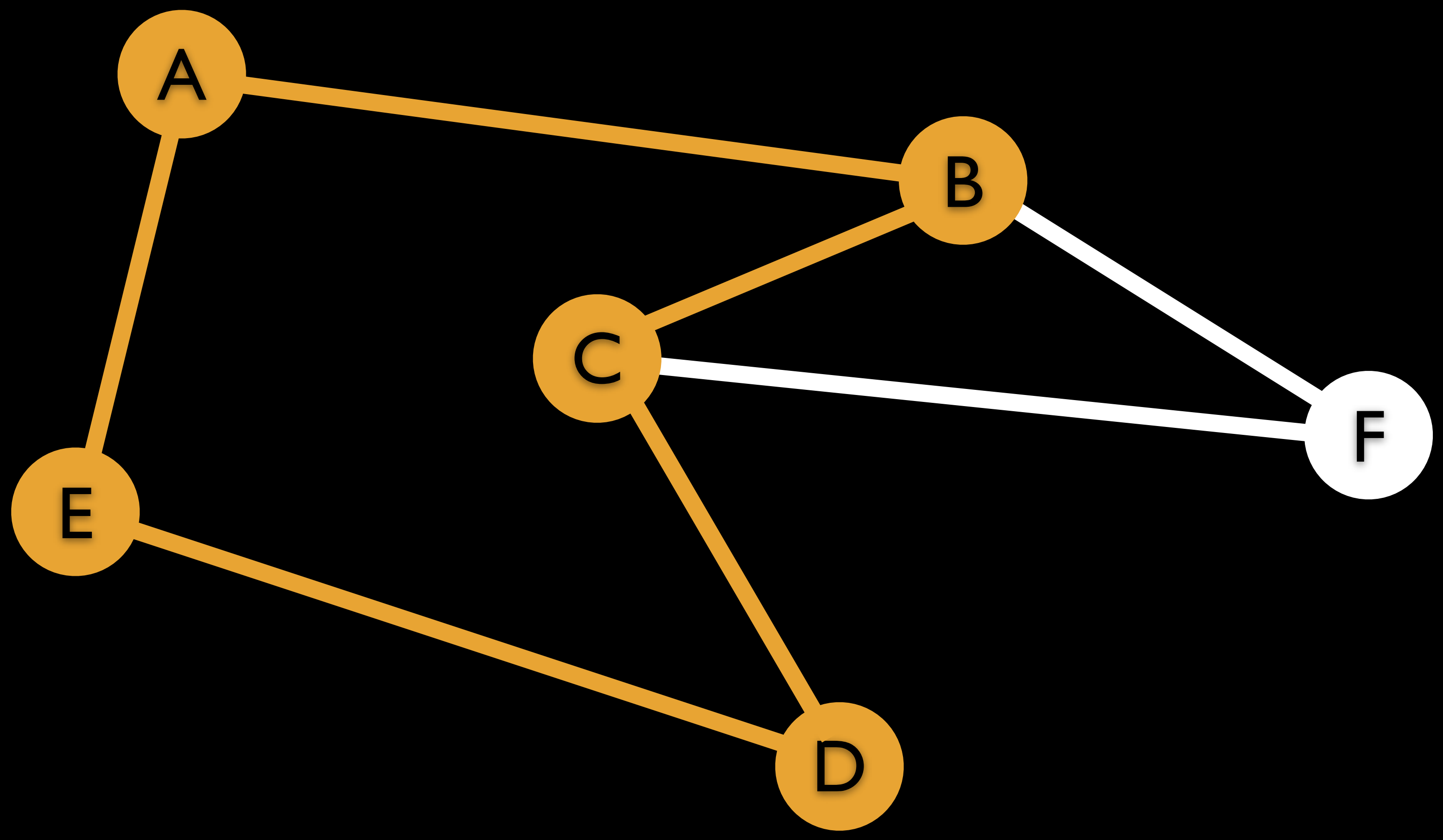
Graphs



Vertices = $\{A, B, C, D, E, F\}$

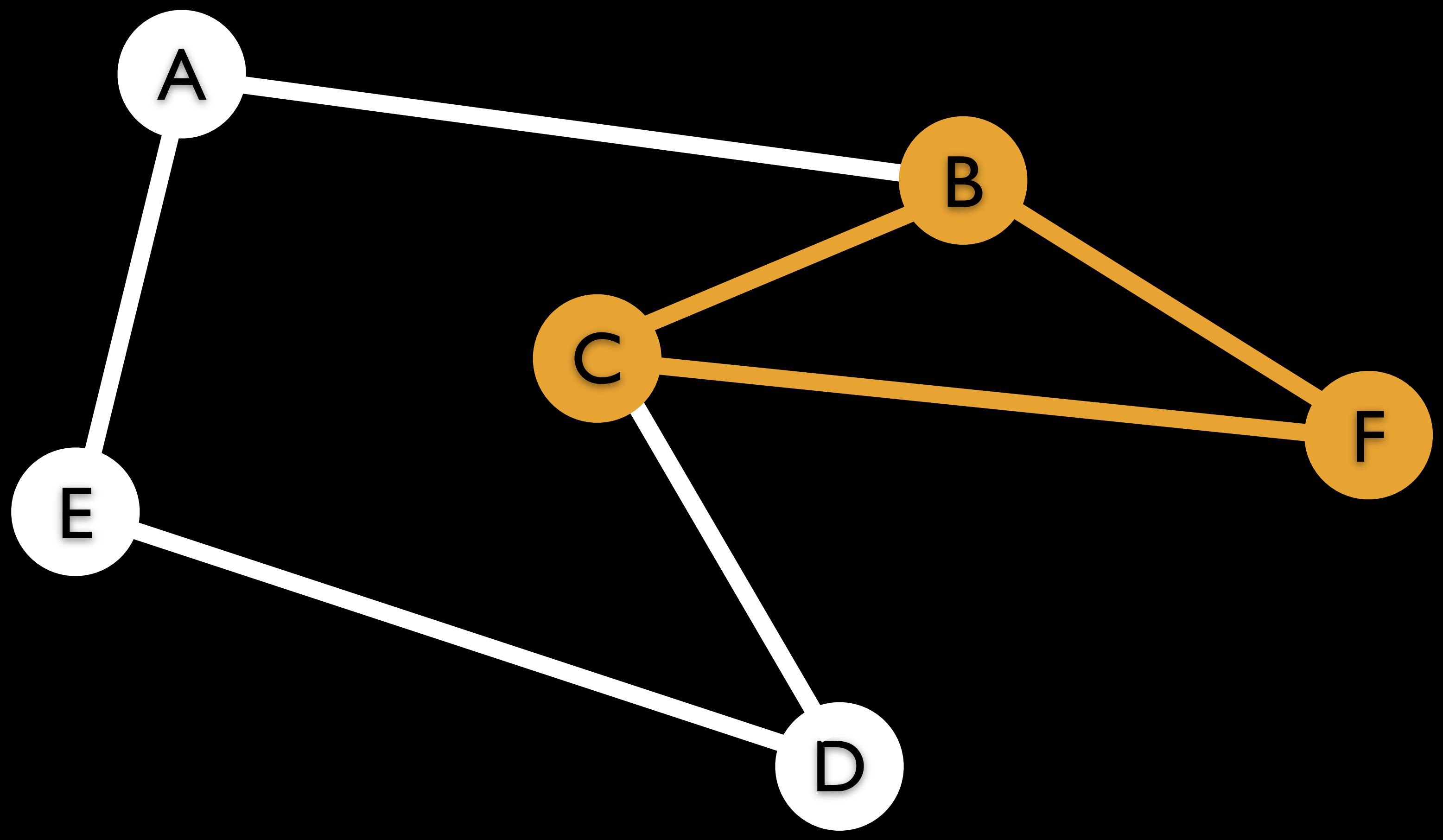
Edges = $\{\{A, B\}, \{A, E\}, \{B, C\}, \{C, D\}, \{C, F\}, \{B, F\}\}$

Graphs



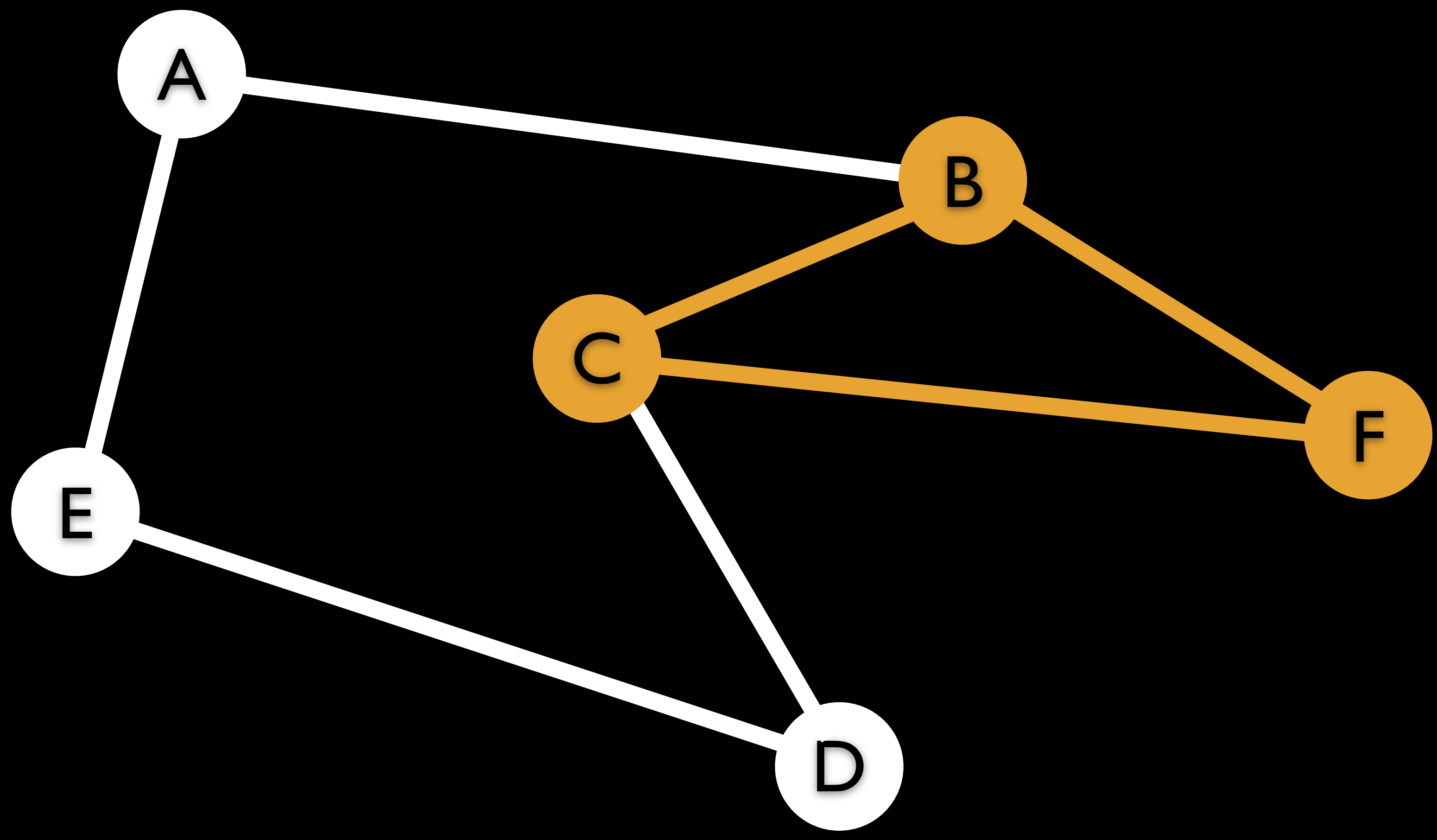
Cycles

Graphs



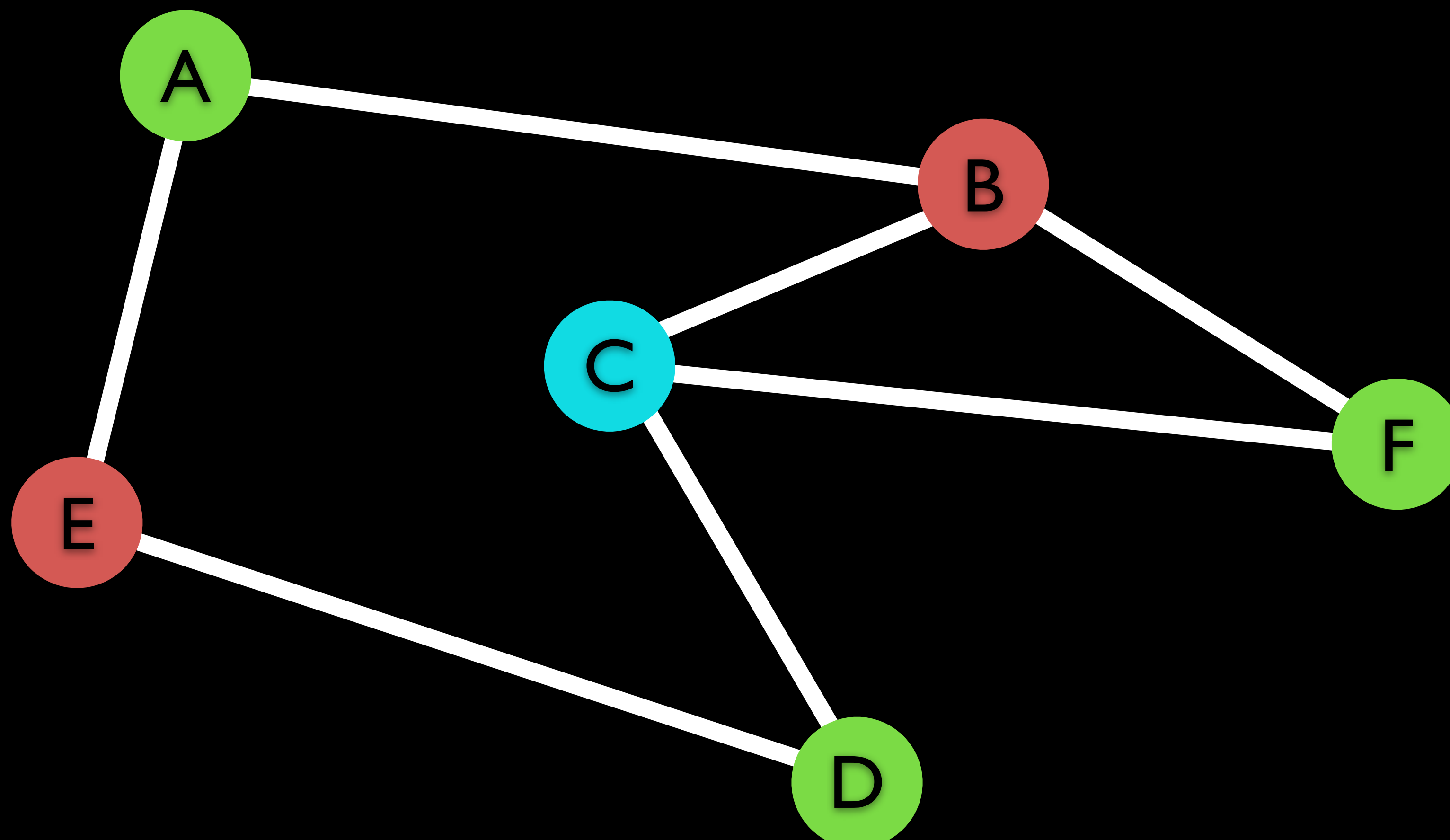
Cycles

Graphs



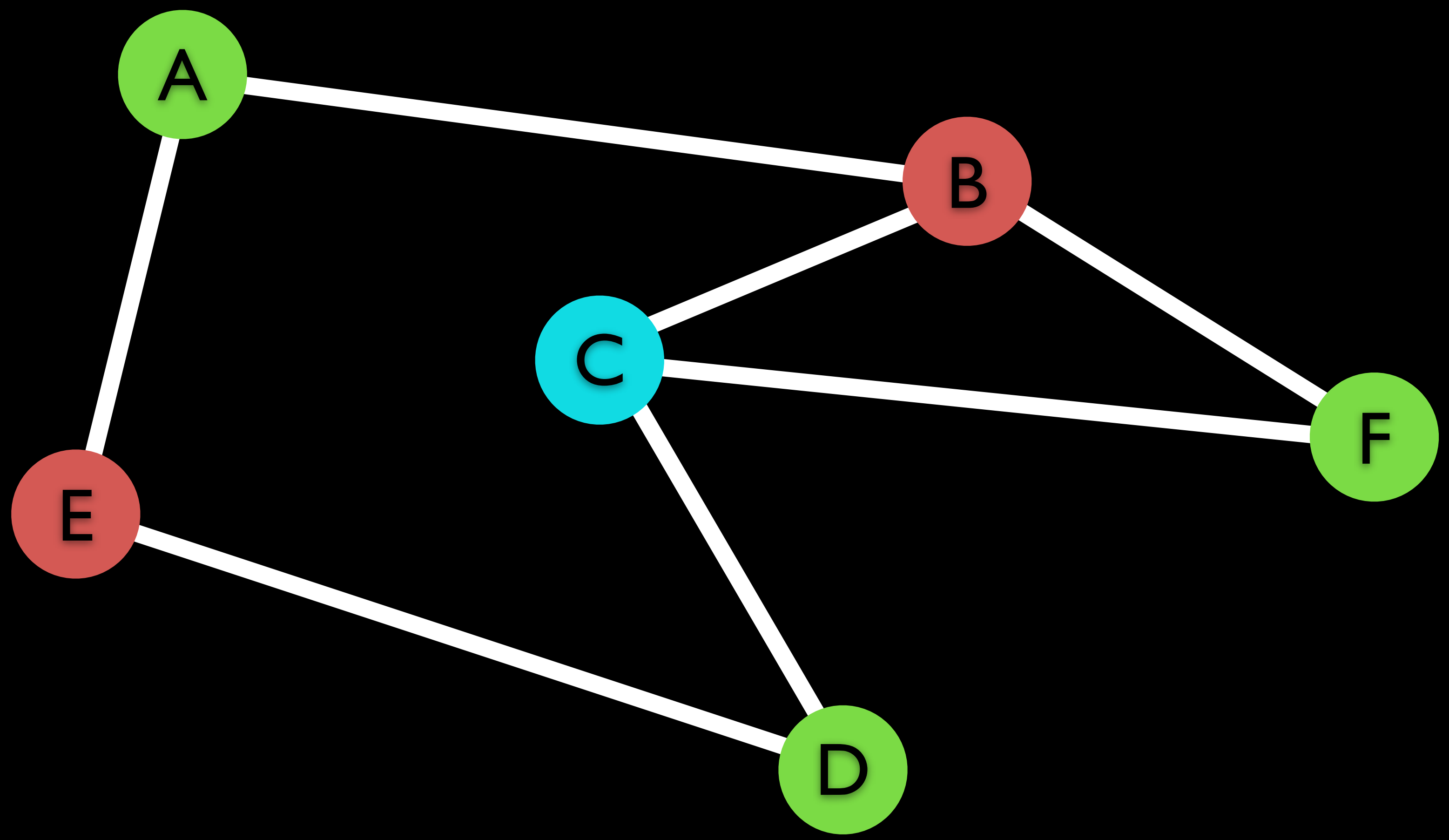
Shortest Cycle = Girth = 3

Graphs



Coloring

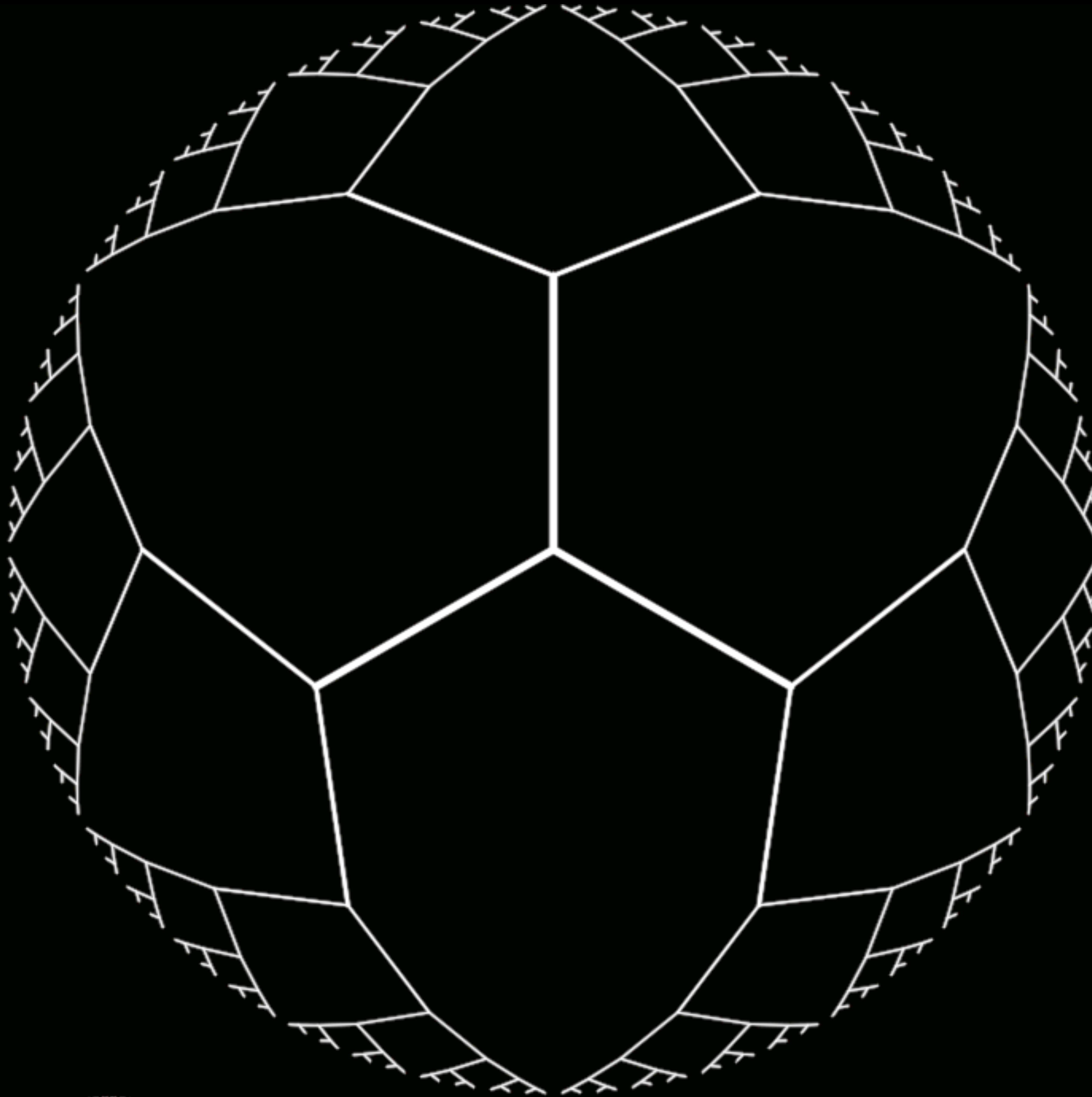
Graphs

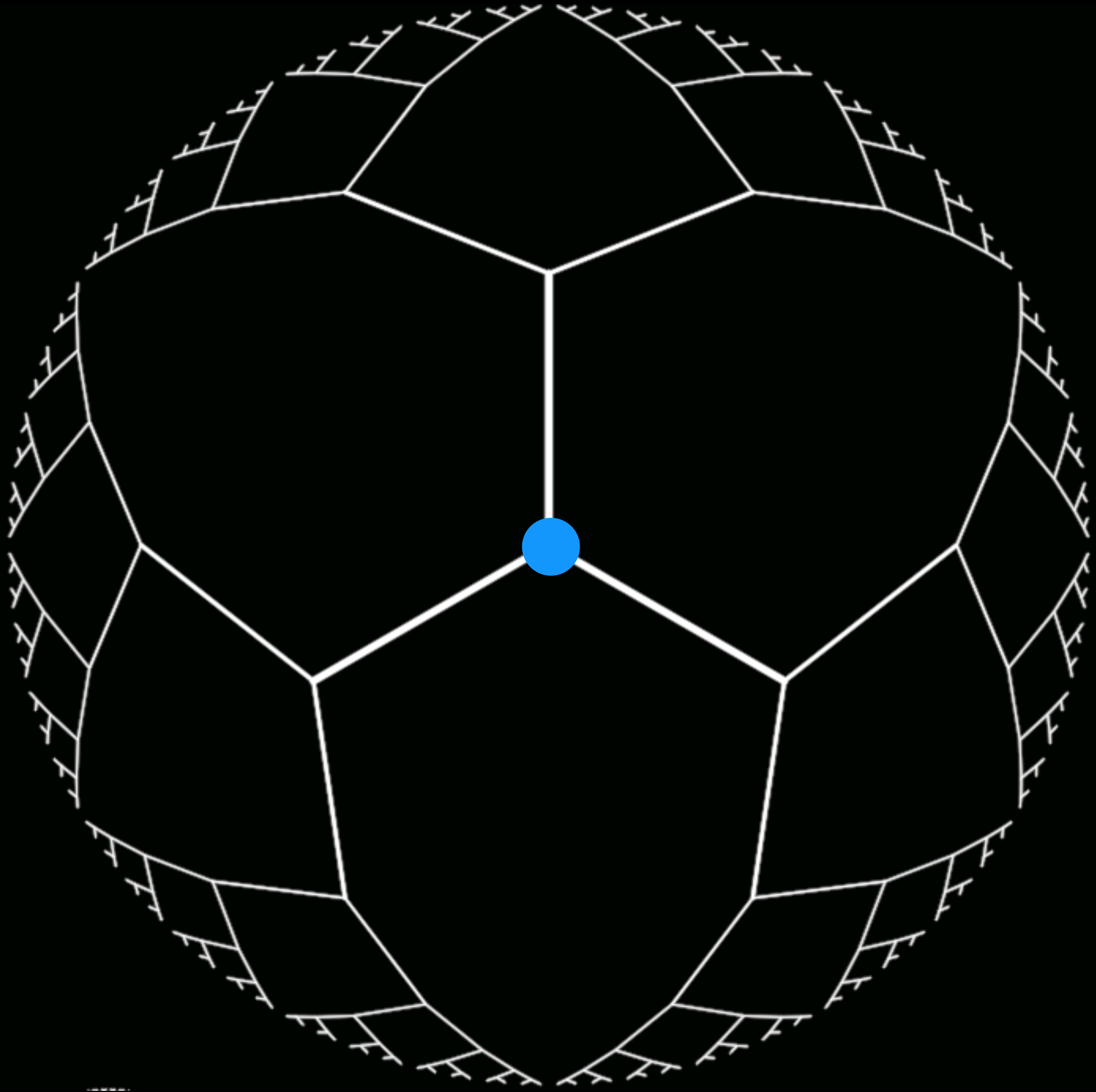


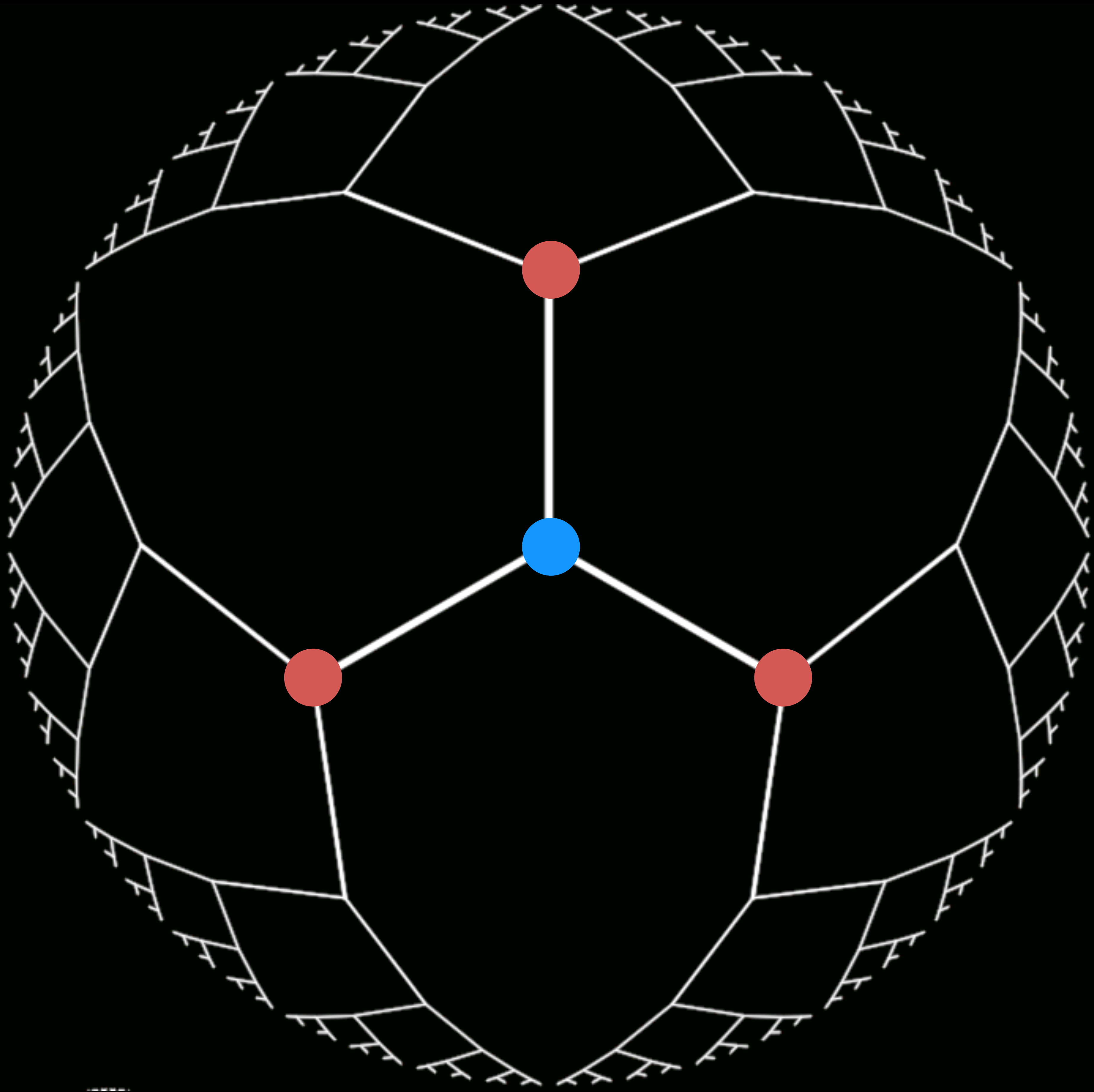
Chromatic Number = 3

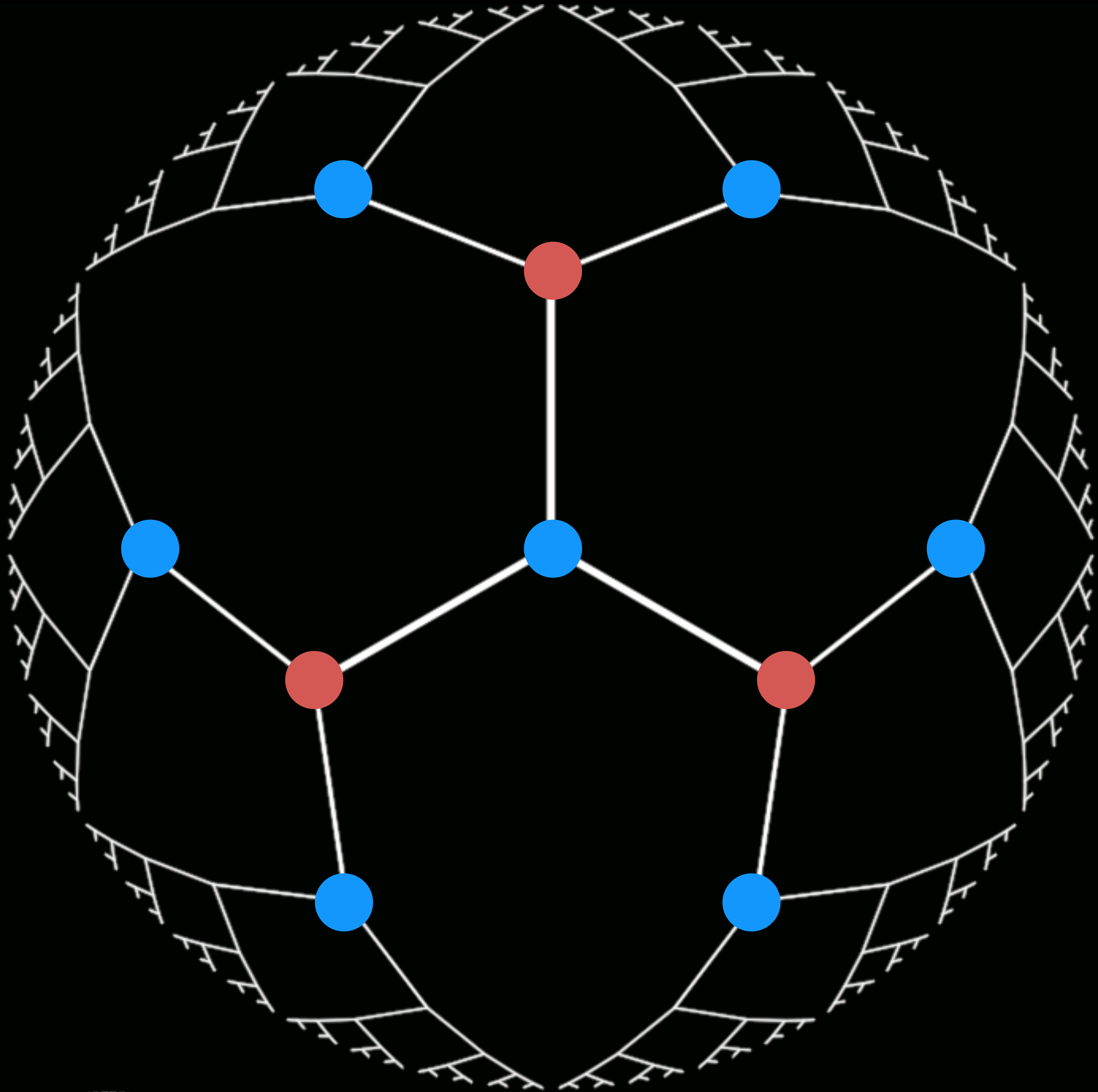
Can a graph have both high girth and large chromatic number?

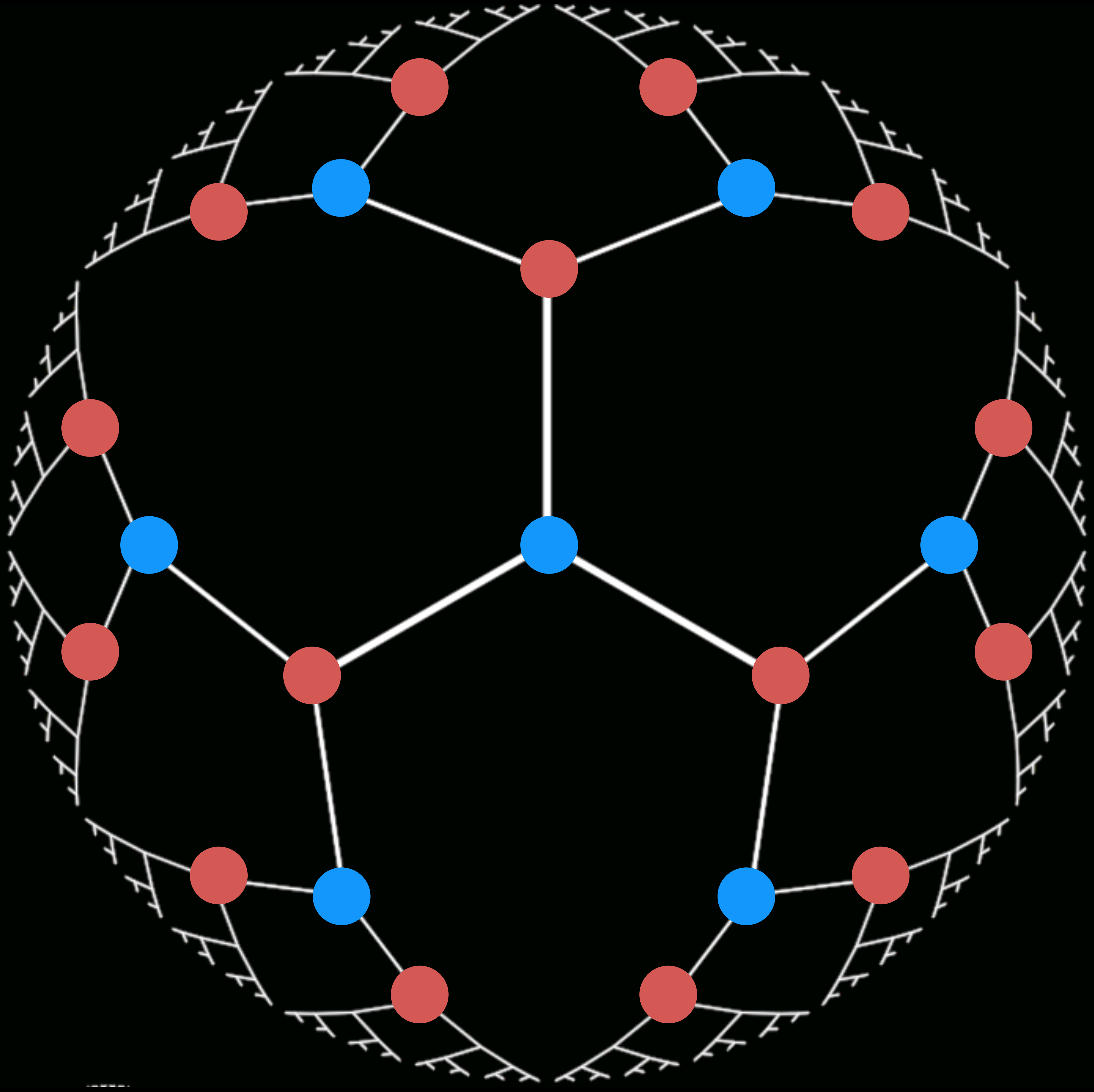
High girth: 2-colorable
locally everywhere.







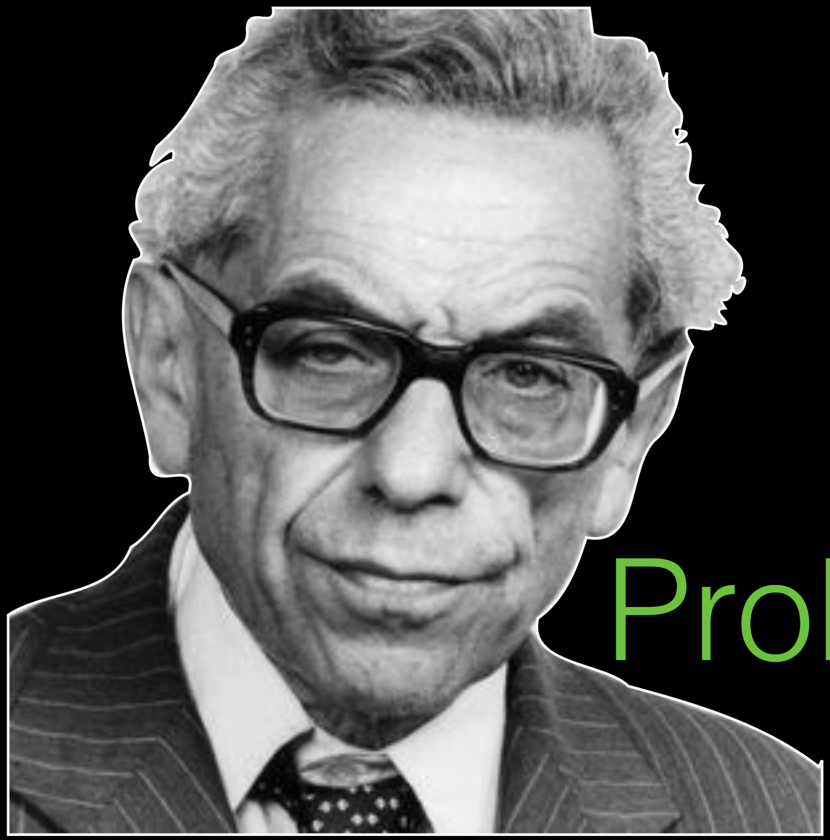




How can we force many colors if every region is 2-colorable?

That's HARD.

Unless…

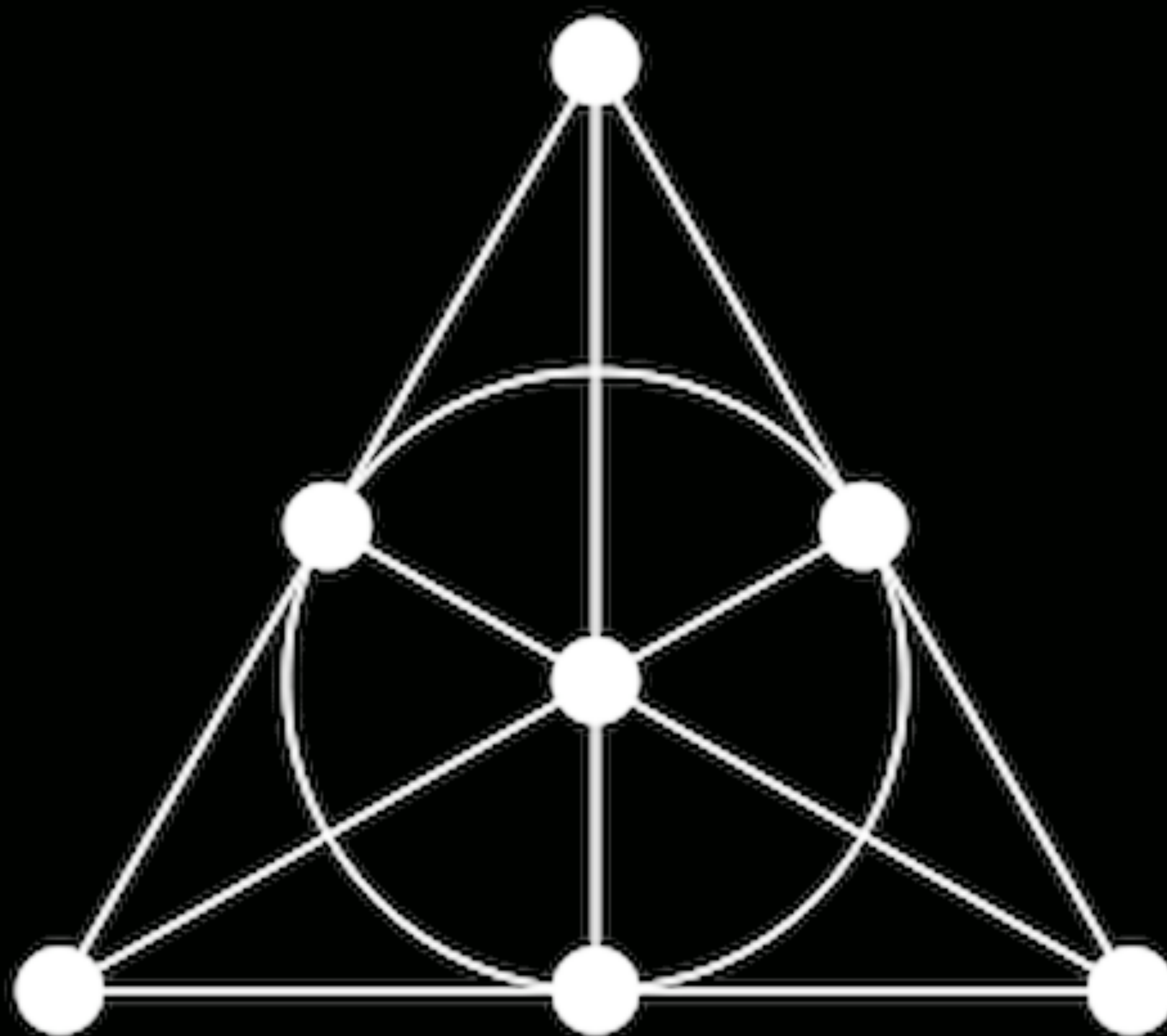


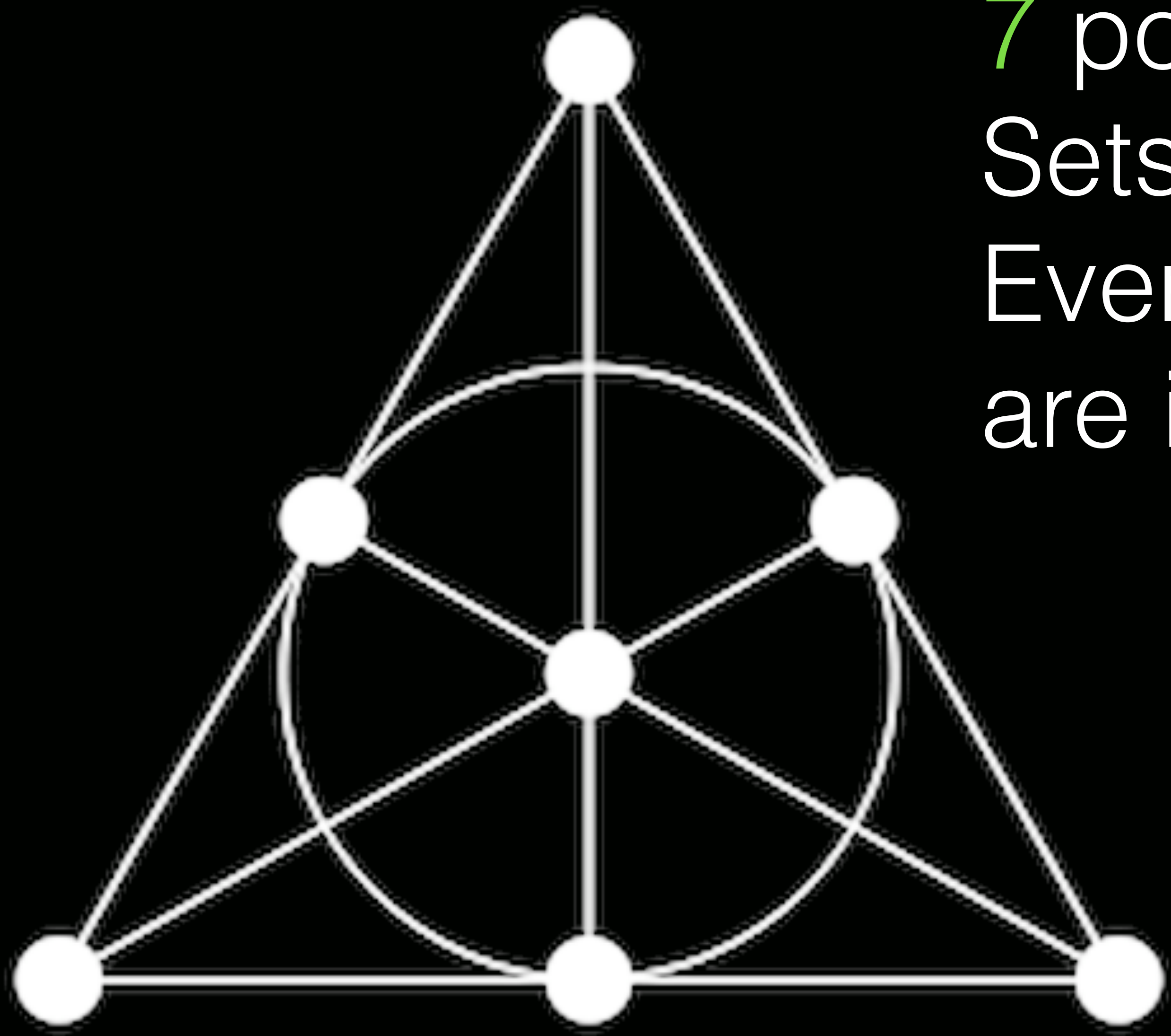
…you use the

Probabilistic Method.

- Take many vertices
- Choose each edge with probability p
- Carefully choose p
- Make the graph have few short cycles and small independence number
- Remove vertices to eliminate short cycles
- Voila!

3. The Existence of Designs





7 points
Sets of size 3
Every 2 points
are in exactly 1 set

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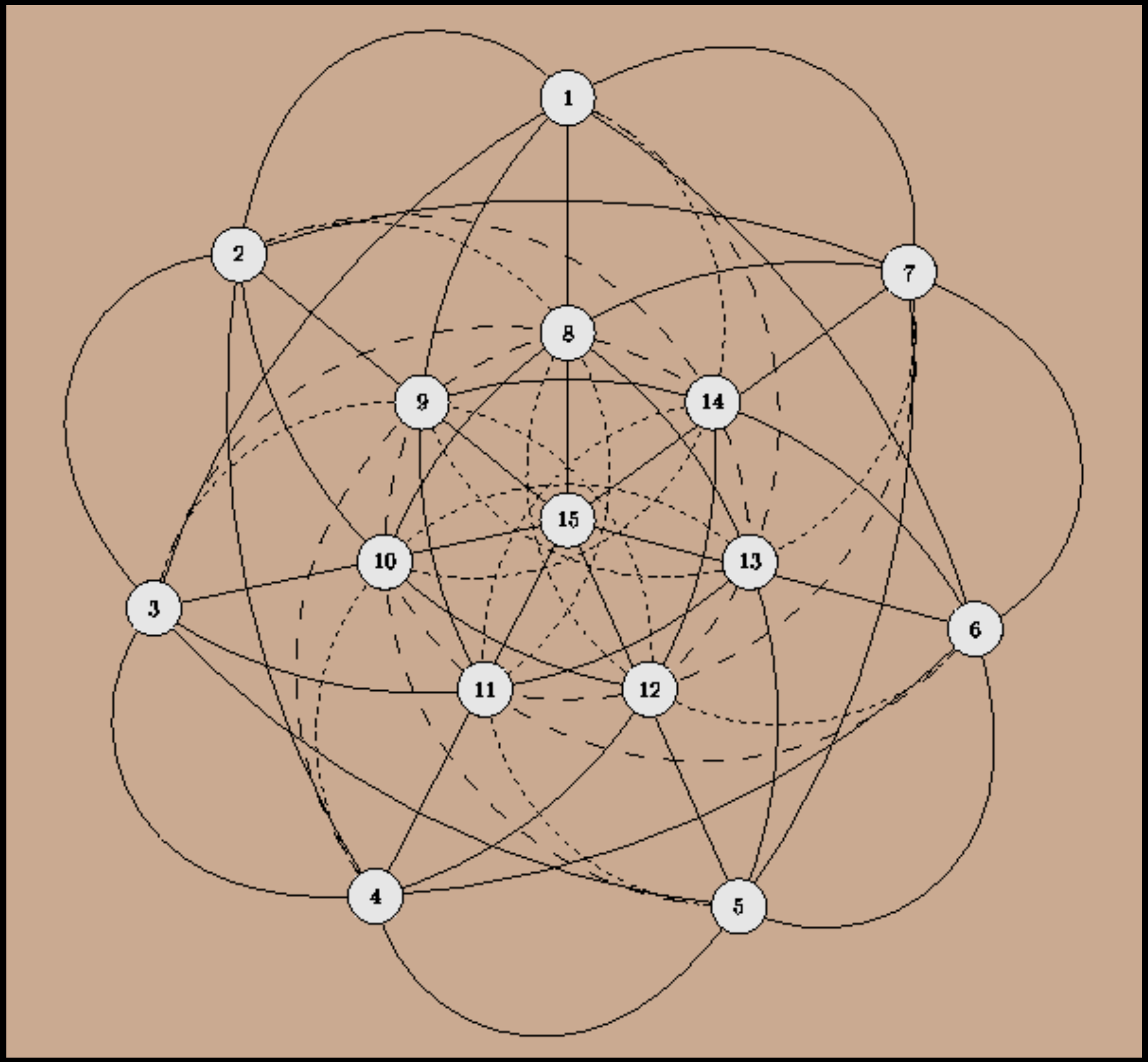


n points

Sets of size *q*

Every *r* points

are in exactly λ sets

n points
Sets of size q
Every r points
are in exactly λ sets

$n\ q\ r\ \lambda$?

$n\ q\ r\ \lambda$?

Divisibility conditions

Finitely many exceptions in n

Asked: 1853

Answered: Jan 15, 2014

Solver: Peter Keevash

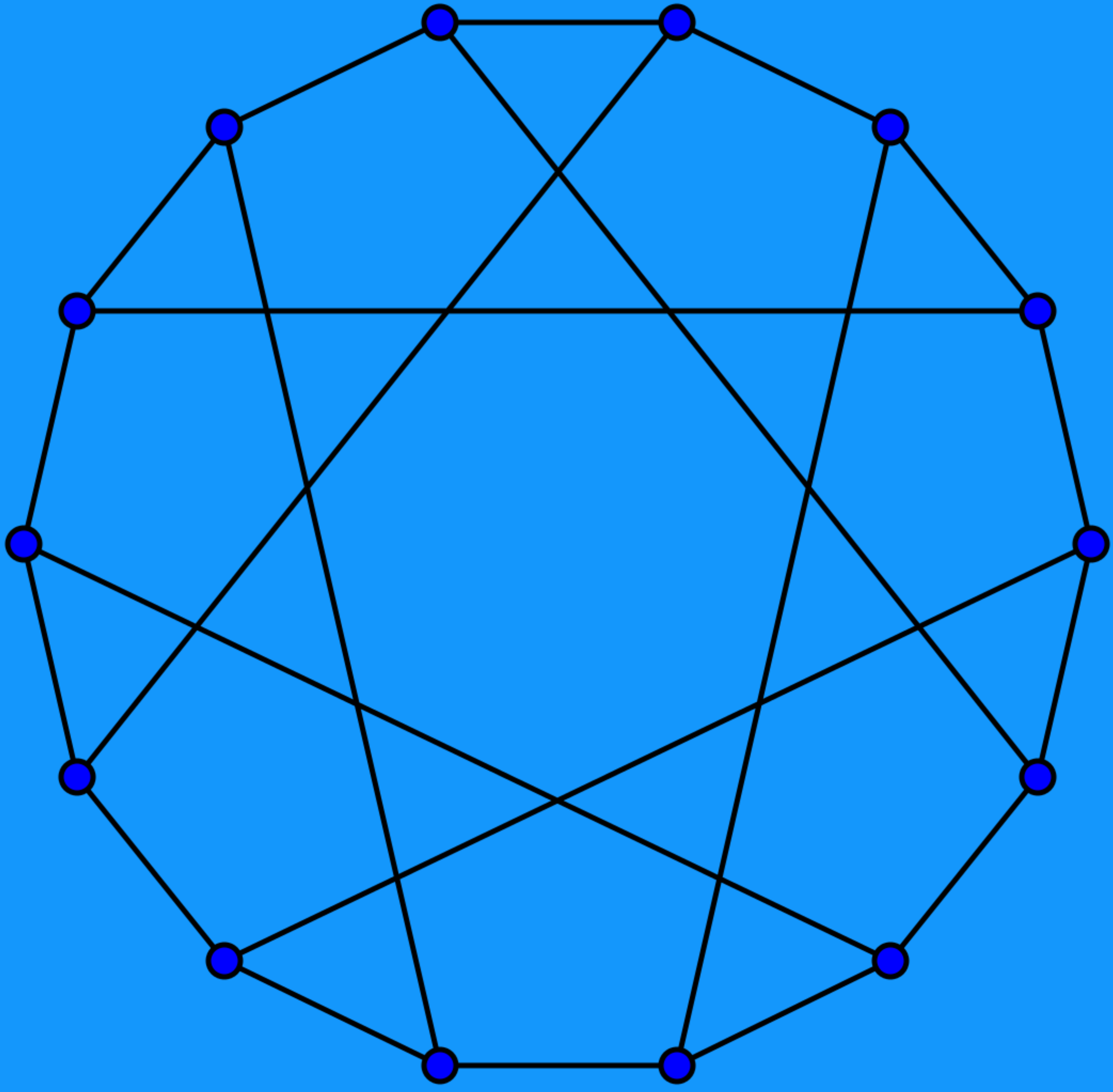
Method: Randomized Algebraic Construction

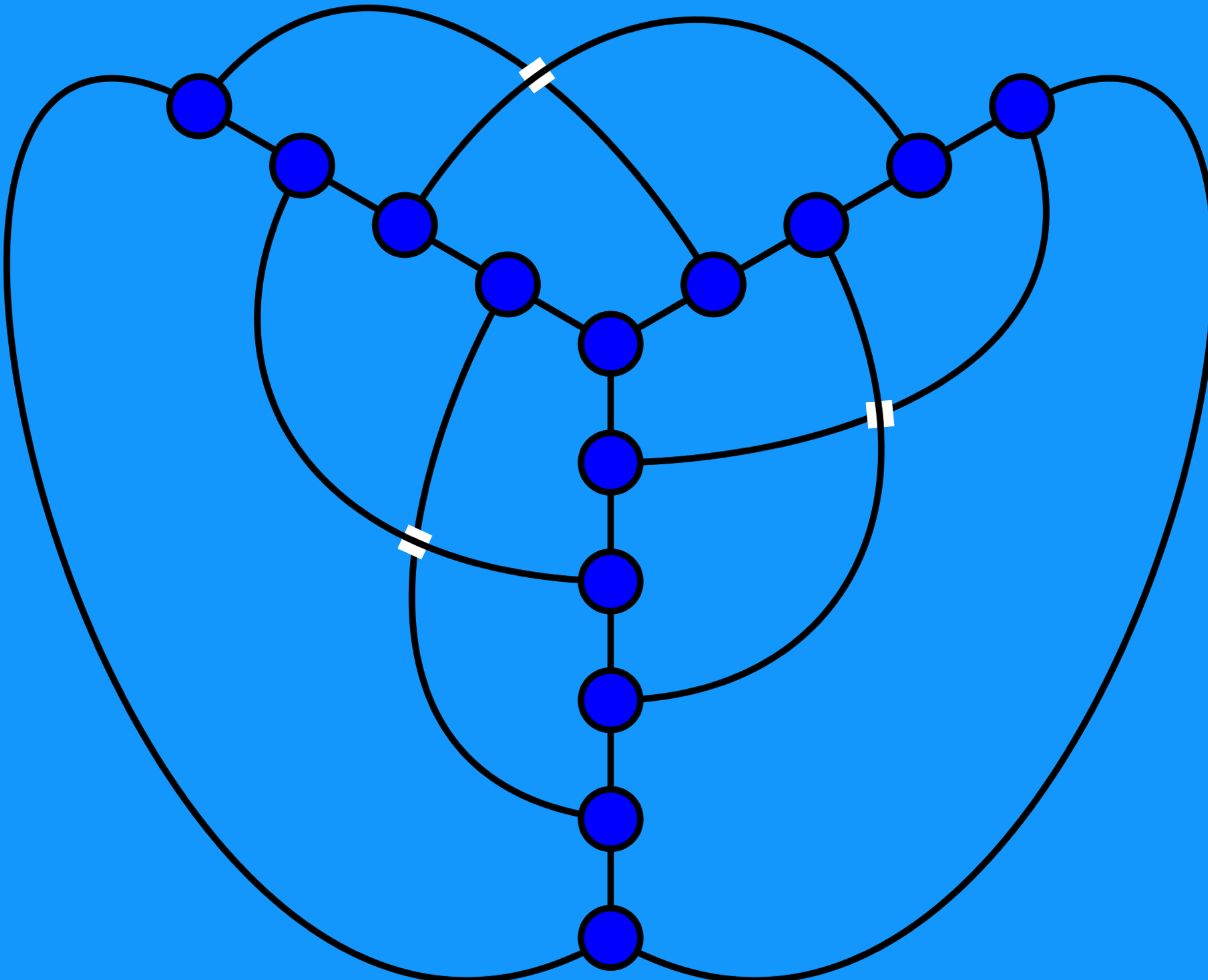
- Rephrase the problem as hypergraph matching
- Seek a matching by randomly picking edges and deleting their overlaps
- The beginning is easy, but the end is out of control
- Keevash: Cleverly pick stand-ins for the end game

4. The Crossing Lemma

Crossing Number:

Minimum number of edge crossings in a plane drawing of a graph





Crossing Lemma: if $e \geq 4v$ then $\mathrm{cr}(G) \geq \dfrac{e^3}{64v^2}$

- Fact (easy): $\mathrm{cr}(G) \geq e - 3v$
- Start with any graph
- Pick a random subgraph H by choosing each vertex with probability p
- Find the expected number of vertices, edges and crossings of H
- Apply the easy fact. Pick the best p. Done!

5. Algorithms

Quicksort

Codes

Primality Testing

Min-Cut

Matrix Testing

Machine Learning

Google™

facebook®

Machine Learning

airbnb™

YAHOO!

ebay™

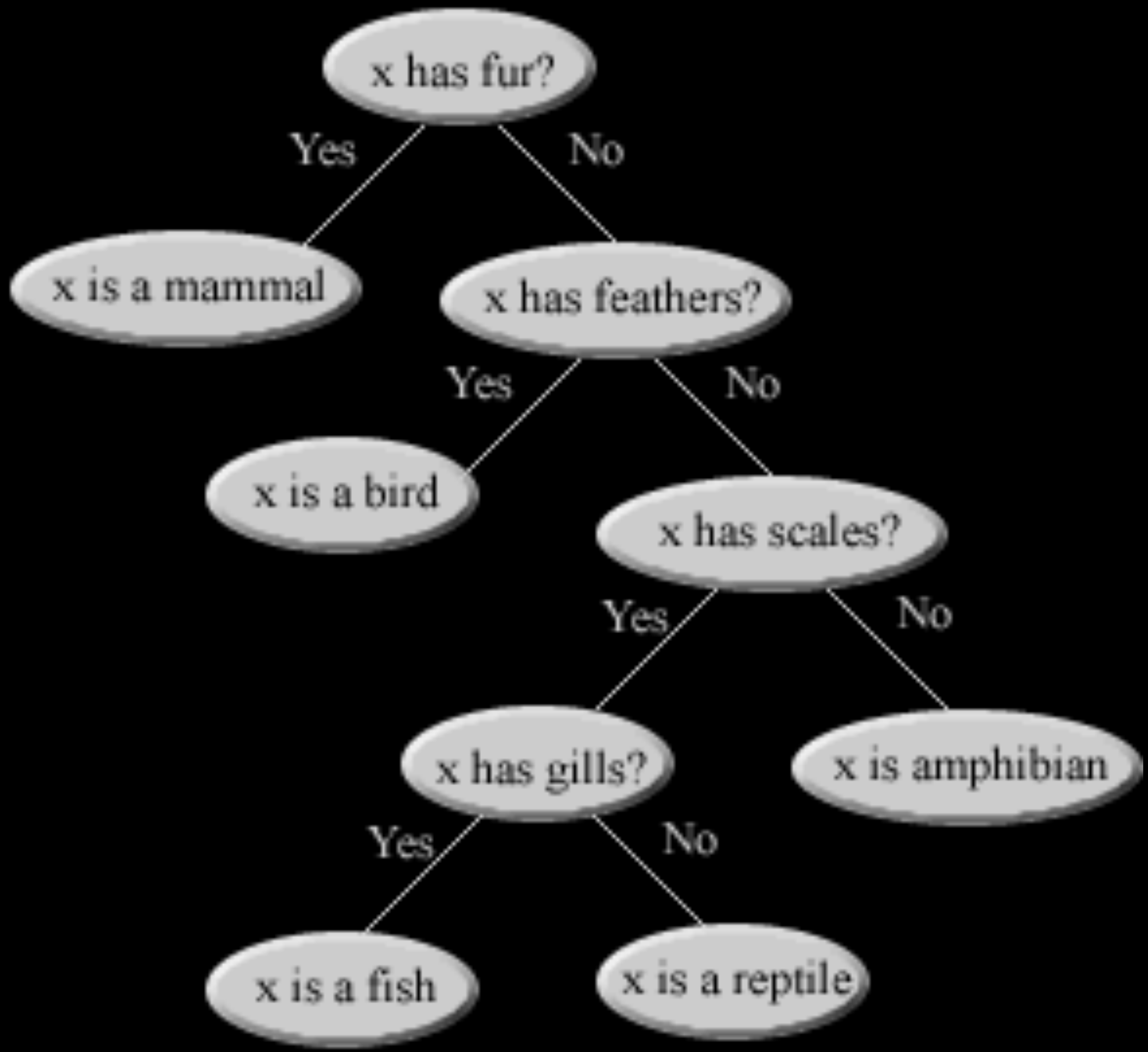
Palantir

Machine Learning

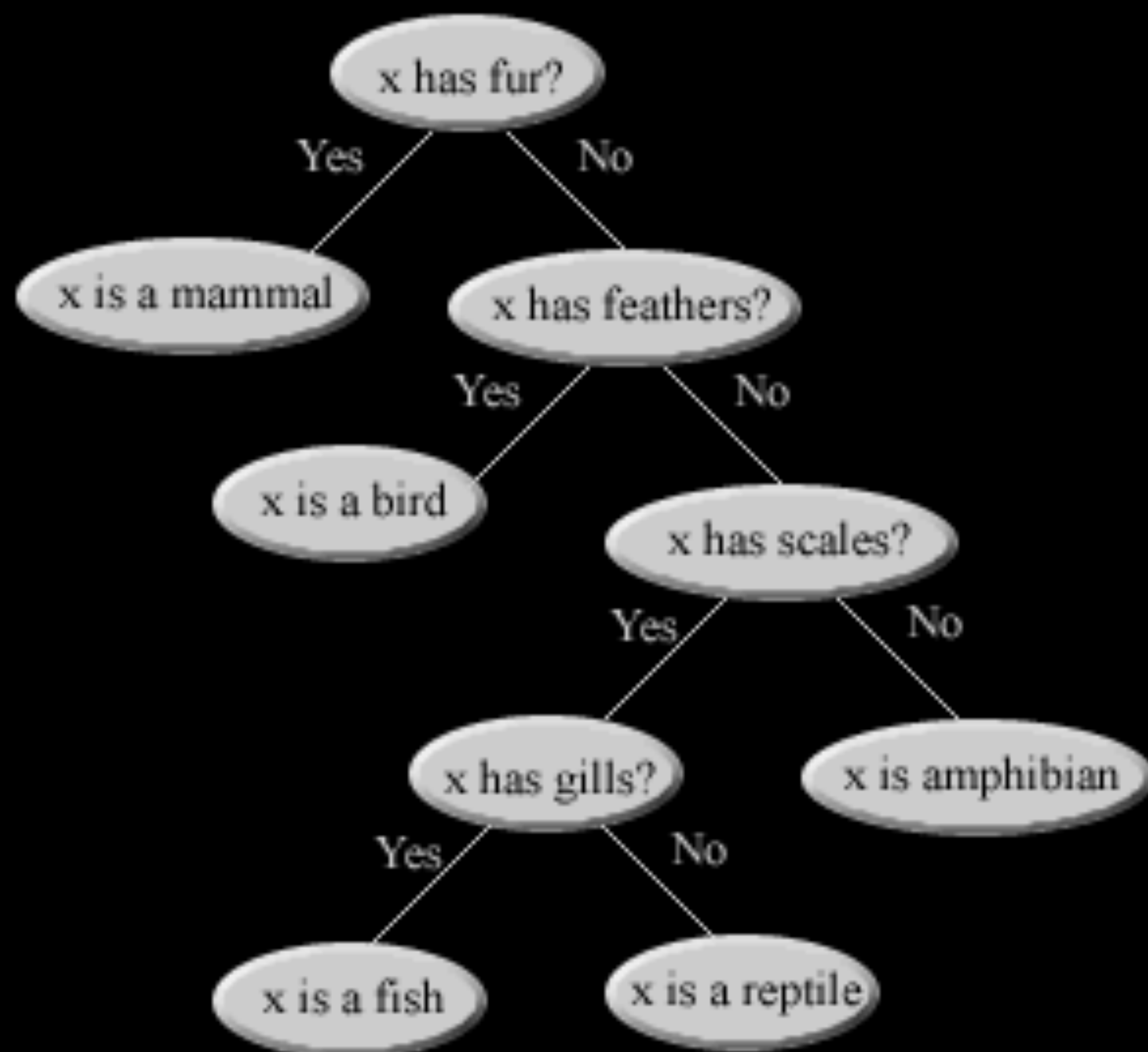
Learn from data: features and outcomes

Predict: Given features, what's the outcome?

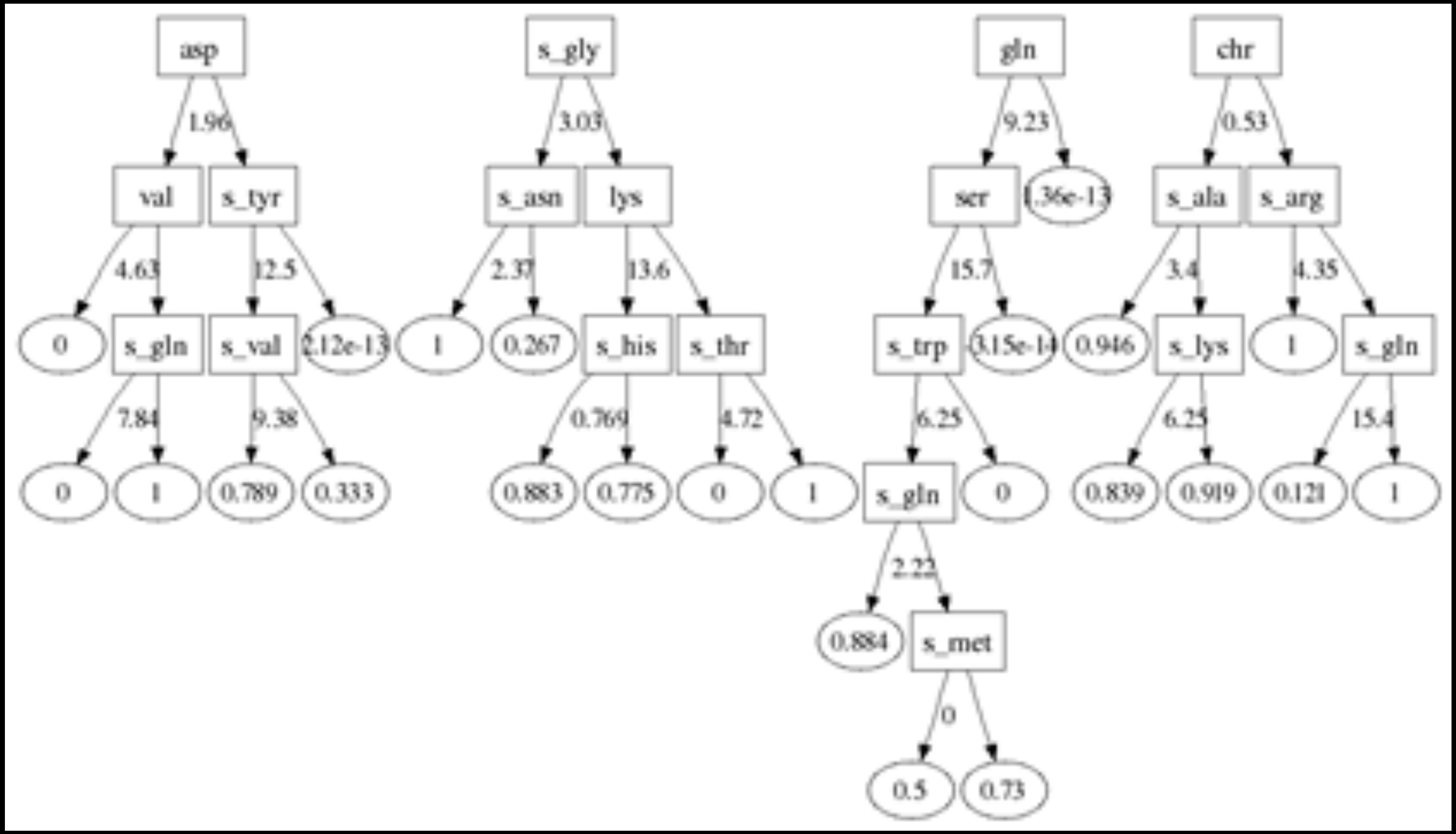
Decision Trees



Decision Trees are great…



…Random Forests are Even Better.



…Random Forests are Even Better.

Train 100 decision trees with random data and random features

Merge into one predictor

Random Forests

Perhaps the single most successful Machine Learning paradigm

Ever.

Summary

Randomness

Mathematics

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Mathematics

Artificial Intelligence